Weighted propositional configuration logic over De Morgan algebras

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Summary

Preliminary Notions

Definition (Lattices)

A **lattice** is a partially ordered set $(K, \leq)$, such that for any $k_1, k_2 \in K$, there exist their supremum $k_1 \oplus k_2$ and infimum $k_1 \otimes k_2$ in K.

- A lattice is called **distributive** if infimum distributes over supremum, i.e.,

$$k_1 \otimes (k_2 \oplus k_3) = (k_1 \otimes k_2) \oplus (k_1 \otimes k_3).$$

- A lattice is called **bounded** if there exist $0, 1 \in K : \forall k \in K, 0 \leq k$ and $k \leq 1$.

Definition (De Morgan Algebras)

A **De Morgan Algebra** $\mathcal{K}$ is a bounded distributive lattice $(K, \leq, \neg)$ equipped with a complement mapping $\neg : K \to K$ which satisfies the involution and De Morgan laws i.e.:

$$\neg(\neg k) = k$$
$$\neg(k_1 \oplus k_2) = \neg k_1 \otimes \neg k_2$$
$$\neg(k_1 \otimes k_2) = \neg k_1 \oplus \neg k_2,$$

for all $k, k_1, k_2 \in K$, where $\oplus$ and $\otimes$ are the supremum and infimum operators in K, respectively.

Preliminary Notions

Proposition (*Droste, Kuich, Rahonis* (2008))

Let $\mathcal{K} = (K, \leq, \neg)$ be a De Morgan algebra. Then $(K, \oplus, \otimes, 0, 1)$ is a commutative semiring with the complement mapping $\neg : K \to K$.

For the rest of this presentation, $(K, \oplus, \otimes, 0, 1, \neg)$ will denote a De Morgan algebra and $\leq$ will denote its partial order.

Example

The standard fuzzy algebra:

$$([0, 1], \max, \min, 0, 1, \neg),$$

where $\neg k := 1 - k$ is the usual subtraction of k from 1 and $\leq$ is the usual real number order, is a De Morgan algebra.

Preliminary Notions

Definition

A **(formal power) series** over Q and K, is a mapping $s : Q \to K$.

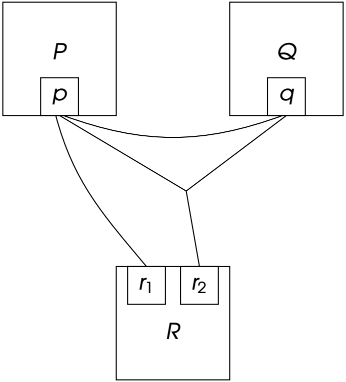
We denote by $K\langle\langle Q\rangle\rangle$ the class of all series over Q and K.

Definition

The **support** of a series s is the set $supp(s) = \{q \in Q \mid s(q) \neq 0\}$.
A **polynomial** is a series with finite support.

We denote by $K\langle Q\rangle$ the set of all polynomials over Q and K.

Components and Ports

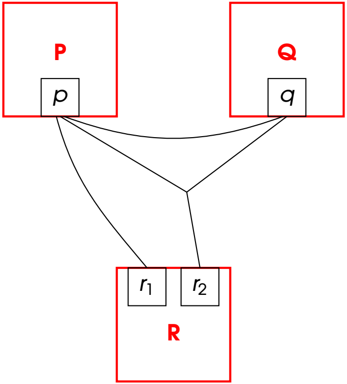


We want to transition from the usual "lines and boxes" representation of architectures to a more mathematically formal form.

- The parts that make up a software architecture can be described with components.
- Each component has a set of ports.
- Ports can interact with each other, connecting the components of a system.

Our goal is to formalize these interactions with a set of logic formulas that correspond to the rules of each architecture.

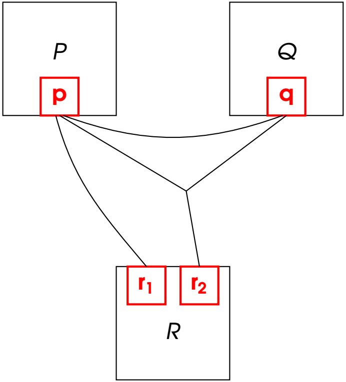
Components, Ports and Interactions



Components

Set of components $\{P, Q, R\}$

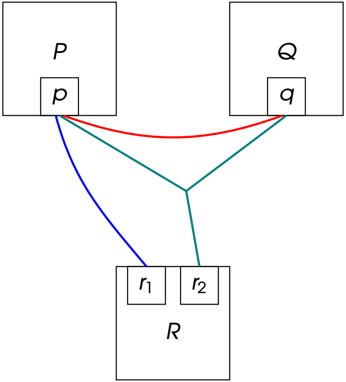
Components, Ports and Interactions



Ports

Set of ports $\{p, q, r_1, r_2\}$

Components, Ports and Interactions



Interactions

We define them as (non-empty) **sets of ports**.

The interactions of this example:

$$\{p, q\}, \{p, r_1\}, \{p, q, r_2\}$$

Syntax and Semantics of PIL

The **Propositional Interaction Logic** is the usual Boolean Logic over the ports of our system, and can sufficiently describe any interaction present in it:

Definition (Syntax of PIL)

The syntax of PIL over P is defined by the grammar:

$$\phi ::= true \mid p \mid \overline{\phi} \mid \phi \vee \phi$$

for any port $p \in P$.
The **Conjunction** operator is defined as ususal: $\phi_1 \wedge \phi_2 := \overline{\overline{\phi_1} \vee \overline{\phi_2}}$

We denote by $\mathbf{I(P)} = 2^P \setminus \{\emptyset\}$ the set of all interactions of a set of ports P.

Syntax and Semantics of PIL

The satisfiability of PIL formulas is defined by interactions as follows:

Definition (Semantics of PIL)

Let ϕ be a PIL formula and $a \in I(P)$ an interaction. The semantics of ϕ is defined inductively as follows:

$$
\begin{aligned}
a &\models_i true && \text{always},\\
a &\models_i p && \text{iff } p \in a,\\
a &\models_i \overline{\phi} && \text{iff } a \not\models_i \phi,\\
a &\models_i \phi_1 \vee \phi_2 && \text{iff } a \models_i \phi_1 \text{ or } a \models_i \phi_2.
\end{aligned}
$$

The semantics of the PIL conjunction as defined above, is the usual:

$$
a \models_i \phi_1 \wedge \phi_2 \qquad \text{iff } a \models_i \phi_1 \text{ and } a \models_i \phi_2.
$$

Monomials

Definition

A **monomial** m is a PIL formula that can be written as a conjunction of ports or their PIL negation, i.e.:

$$\exists P_+, P_- \subseteq P : m = \bigwedge_{p \in P_+} p \wedge \bigwedge_{p \in P_-} \overline{p}$$

where $P_+ \cap P_- = \emptyset$.
If additionally $P_+ \cup P_- = P$ then m is called a **full monomial**.

When describing monomials, we will omit the conjunction operators, for example, $p_1\overline{p_2}p_3p_4$ will represent the monomial $p_1 \wedge \overline{p_2} \wedge p_3 \wedge p_4$.

Example

Let $P = \{p, q, r\}$. Then pq, $\overline{p}r$ and $p\overline{q}r$ are monomials, but, amongst them, only $p\overline{q}r$ is a full monomial.

Monomials

Definition

Let $a \in I(P)$. We call **characteristic monomial** of a the following full monomial:

$$m_a = \bigwedge_{p \in a} p \wedge \bigwedge_{p \notin a} \overline{p}$$

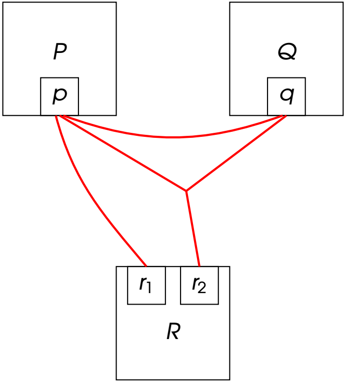
such that for any $a' \in I(P)$, $a' \models_i m_a$ iff $a' = a$.

Example

Let $P = \{p, q, r, s\}$. The characteristic monomial of $a = \{p, s\} \in I(P)$ is:

$$m_a = p\, s\, \overline{q}\, \overline{r}$$

Configurations



Configurations

We define them as (non-empty) **sets of interactions**, i.e., sets of sets of ports.

The configuration of this example:

$$\gamma = \{\{p, q\}, \{p, r_1\}, \{p, q, r_2\}\}$$

Syntax and Semantics of PCL

The **Propositional Configuration Logic** is an extension of PIL that has the additional operators of Union $\sqcup$, Coalescing $+$ and Complementation $\neg$.

Definition (Syntax of PCL)

The syntax of PCL over P is defined by the grammar:

$$f ::= true \mid \phi \mid \neg f \mid f + f \mid f \sqcup f$$

where ϕ is a PIL formula.

We can also define the usual **intersection** and **implication** operators:

$$f_1 \sqcap f_2 := \neg(\neg f_1 \sqcup \neg f_2)$$

$$f_1 \Rightarrow f_2 := \neg f_1 \sqcup f_2$$

Mavridou, Baranov, Bliudze, Sifakis: Configuration logics: modeling architecture styles (2017)

We denote by $C(P) = 2^{I(P)} \setminus \{\emptyset\}$ the set of all configurations of P.

Syntax and Semantics of PIL

The satisfiability of PCL formulas is defined by configurations as follows:

Definition (Semantics of PCL)

Let P be a set of ports and $\gamma \in C(P)$ be a configuration. The semantics of γ are defined inductively as follows:

$$\begin{aligned}
&\gamma \models \mathit{true} && \text{always} \\
&\gamma \models \phi && \text{iff } \forall a \in \gamma, a \models_i \phi, \text{ where } \phi \text{ is an interaction formula} \\
&\gamma \models f_1 + f_2 && \text{iff } \exists \gamma_1, \gamma_2 \in C(P) : \gamma = \gamma_1 \cup \gamma_2, \gamma_1 \models f_1 \text{ and } \gamma_2 \models f_2 \\
&\gamma \models f_1 \sqcup f_2 && \text{iff } \gamma \models f_1 \text{ or } \gamma \models f_2 \\
&\gamma \models \neg f && \text{iff } \gamma \not\models f
\end{aligned}$$

The semantics of the intersection and implication operators are the usual:

$$\begin{aligned}
&\gamma \models f_1 \sqcap f_2 && \text{iff } \gamma \models f_1 \text{ and } \gamma \models f_2 \\
&\gamma \models f_1 \Rightarrow f_2 && \text{iff } \gamma \not\models f_1 \text{ or } \gamma \models f_2
\end{aligned}$$

Lastly, we define *false* as usual: $\mathit{false} := \neg \mathit{true}$, and we have $\forall \gamma \in C(P) : \gamma \not\models \mathit{false}$.

Syntax and Semantics of PCL

Definition (Equivalence of PCL formulas)

Two PCL formulas f_1, f_2 will be called **equivalent**, denoted by $f_1 \equiv f_2$, if $\forall \gamma \in C(P) : \gamma \models f_1$ iff $\gamma \models f_2$.

Definition

The **characteristic configuration set** of a PCL formula f is defined as the set $|f| := \{\gamma \in C(P) \mid \gamma \models f\}$.

It is clear from the above definitions that for two PCL formulas f_1, f_2 we have $f_1 \equiv f_2$ iff $|f_1| = |f_2|$.

Proposition (Intersection ≡ Disjunction)

For any interaction formulas ϕ_1, ϕ_2, it holds that $\phi_1 \wedge \phi_2 \equiv \phi_1 \sqcap \phi_2$ and $\phi_1 + \phi_1 \equiv \phi_1$.

Therefore, from now on, for both the PIL disjunction and PCL intersection operators, we will use the $\wedge$ symbol. We also conservatively extend the PIL conjunction to PCL with the following semantics for any PCL formulas f_1, f_2

$$\gamma \models f_1 \vee f_2 \text{ iff } \gamma \models f_1 \sqcup f_2 \sqcup f_1 + f_2.$$

Syntax and Semantics of PCL

Definition

The **closure operator** $\sim$ of a PCL formula f is defined as follows:

$$\sim f := f + true$$

and its respective semantics is:

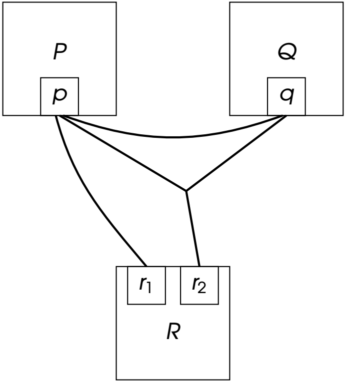
$$\gamma \models \sim f \text{ iff } \exists \gamma' \subseteq \gamma : \gamma' \models f$$

Lemma

A formula $f = \sum_{i \in I} m_i$, where m_i are full monomials, is satisfied by exactly one configuration $\gamma = \{a_i \mid i \in I\}$ where for all $i \in I$, a_i is the unique interaction such that $a_i \models_i m_i$.

$\sum_{i \in I} m_i$ denotes the coalescing of the monomials $m_i, i \in I$.
The reverse of the above lemma holds, i.e., for every configuration there exists a coalescing of full monomials such that only γ satisfies that formula.

Formalisation of Interactions and Configurations



In this example the displayed interactions can be represented by their characteristic monomials:

$\{p, q\}$	by	$p\, q\, \overline{r_1}\, \overline{r_2}$
$\{p, r_1\}$	by	$p\, \overline{q}\, r_1\, \overline{r_2}$
$\{p, q, r_2\}$	by	$p\, q\, \overline{r_1}\, r_2$

The configuration $\gamma = \{\{p, q\}, \{p, r_1\}, \{p, q, r_2\}\}$ can be represented by the PCL formula

$$p\, q\, \overline{r_1}\, \overline{r_2} + p\, \overline{q}\, r_1\, \overline{r_2} + p\, q\, \overline{r_1}\, r_2$$

which is the coalescing of the full monomials above.

Normal Form of PCL formulas

Definition

A PCL formula f is said to be in **normal form** if it is expressed as:

$$f = \bigsqcup_{i \in I} \sum_{j \in J_i} \bigvee_{k \in K_{i,j}} m_{i,j,k},$$

where $I, J_i, K_{i,j}$ are (finite) index sets and $m_{i,j,k}$ are monomials.
f is said to be in **full normal form** if it is expressed as:

$$f = \bigsqcup_{i \in I} \sum_{j \in J_i} m_{i,j},$$

where I, J_i are (finite) index sets and $m_{i,j}$ are **full** monomials.

Proposition

For any PCL formulas f_1, f_2 it holds that:

$$|f_1 \sqcup f_2| = |f_1| \cup |f_2|.$$

Normal Form of PCL formulas

We present 5 rewriting rules for PCL formulas:

PCL Rewriting System

$$g \wedge \bigsqcup_{i \in I} f_i \xrightarrow{\text{Rule 1}} \bigsqcup_{i \in I} g \wedge f_i$$

$$g + \bigsqcup_{i \in I} f_i \xrightarrow{\text{Rule 2}} \bigsqcup_{i \in I} g + f_i$$

$$\neg \bigsqcup_{i \in I} f_i \xrightarrow{\text{Rule 3}} \bigwedge_{i \in I} \neg f_i$$

$$\neg \sum_{\phi \in \Phi} \phi \xrightarrow{\text{Rule 4}} \bigsqcup_{\phi \in \Phi} \overline{\phi} \sqcup \sim \left(\bigwedge_{\phi \in \Phi} \overline{\phi} \right)$$

$$\sum_{\phi \in \Phi} \phi \wedge \sum_{\psi \in \Psi} \psi \xrightarrow{\text{Rule 5}} \sum_{\xi \in \Phi \cup \Psi} \left(\xi \wedge \bigvee_{\phi \in \Phi, \psi \in \Psi} (\phi \wedge \psi) \right)$$

f_i, g are PCL formulas, while ϕ and ψ are PIL formulas.
We also include the usual Boolean transformations and absorption laws.

Normal Form of PCL formulas

Theorem (*Mavridou, Baranov, Bliudze, Sifakis* (2017))

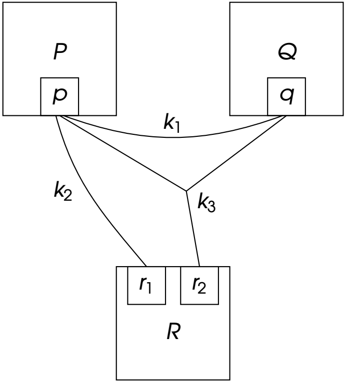
The next three statements about the calculation of the equivalent normal form of a formula hold true:

1. The rewriting system is terminating and confluent (when we only apply rules on formulas whose sub-formulas has no applicable rule)
2. For any formula f' derived from a formula f by the application of a rewriting rule, we have $f' \equiv f$.
3. Any irreducible formula is in normal form.

Theorem (*Paraponiari, Rahonis* (2017))

Let P be a set of ports. Then, for every PCL formula f over P we can effectively construct, in **doubly exponential time**, an equivalent PCL formula f' in full normal form which is unique up to equivalence relation. The best run time for the construction of f' is exponential.

Quantitative Aspects of Architectures



$$p\,\overline{q}\,r_1\,\overline{r_2} + p\,q\,\overline{r_1}\,\overline{r_2} + p\,q\,\overline{r_1}\,r_2$$

PCL is sufficient for the description of most qualitative characteristics of software architectures. However, for most systems to function correctly, there exist several quantitative requirements, such as a time limit, a certain cost or gain for each interaction, the probability of an interaction's implementation etc. In order to keep track of such characteristics in our formulas we will extend PCL such as each formula of the new logic will have a corresponding weight of a De Morgan algebra.

Syntax and Semantics of w_{DM}PCL

Definition (Syntax of w_{DM}PCL)

The syntax of w_{DM}PCL formulas over a non-empty set of ports P and a De Morgan algebra $\mathcal{K} = (K, \oplus, \otimes, 0, 1, \neg)$ is given by the grammar:

$$\zeta ::= k \mid f \mid \zeta \oplus \zeta \mid \zeta \otimes \zeta \mid \neg\zeta \mid \zeta \uplus \zeta$$

where $k \in K$, f is a PCL formula and $\uplus$ is the coalescing operator among w_{DM}PCL formulas.

We denote by $PCL(\mathcal{K}, P)$ the set of w_{DM}PCL formulas over the set of ports P and the De Morgan algebra $\mathcal{K}$. The semantics of w_{DM}PCL formulas $\zeta \in PCL(\mathcal{K}, P)$ will be represented by polynomials $\|\zeta\| \in K\langle C(P)\rangle$.

Syntax and Semantics of w_{DM}PCL

Definition (Semantics of w_{DM}PCL)

Let P be a non-empty set of ports and $\mathcal{K}$ a De Morgan algebra. For every $\zeta \in PCL(\mathcal{K}, P)$, the semantics of ζ is a polynomial $\|\zeta\| \in K\langle C(P)\rangle$ where for every configuration $\gamma \in C(P)$ the value of $\|\zeta\|(\gamma)$ is defined inductively as follows:

$$\|k\|(\gamma) = k$$

$$\|f\|(\gamma) = \begin{cases} 1 & \text{if } \gamma \models f \\ 0 & \text{otherwise} \end{cases}$$

$$\|\zeta_1 \oplus \zeta_2\|(\gamma) = \|\zeta_1\|(\gamma) \oplus \|\zeta_2\|(\gamma)$$

$$\|\zeta_1 \otimes \zeta_2\|(\gamma) = \|\zeta_1\|(\gamma) \otimes \|\zeta_2\|(\gamma)$$

$$\|\neg\zeta\|(\gamma) = \neg\|\zeta\|(\gamma)$$

$$\|\zeta_1 \uplus \zeta_2\|(\gamma) = \bigoplus_{\gamma=\gamma_1 \cup \gamma_2} (\|\zeta_1\|(\gamma_1) \otimes \|\zeta_2\|(\gamma_2))$$

where $k \in K$ and f is a PCL formula and $\cup$ denotes the usual set-union among non-empty sets.

Syntax and Semantics of w_{DM}PCL

Proposition

Two w_{DM}PCL formulas ζ, ξ are called equivalent when their respective polynomials are equal, i.e.,

$$\zeta \equiv \xi \text{ iff } \|\zeta\| = \|\xi\| \text{ iff } \forall \gamma \in C(P) : \|\zeta\|(\gamma) = \|\xi\|(\gamma)$$

Definition (Weighted Closure Operator)

For any w_{DM}PCL formula ζ we define its weighted closure operator as follows:

$$\sim \zeta := \zeta \uplus 1.$$

The respective semantics is:

$$\| \sim \zeta\|(\gamma) = \bigoplus_{\emptyset \neq \gamma' \subseteq \gamma} \|\zeta\|(\gamma')$$

Proposition (w_{DM}PCL closure ≡ PCL closure)

Let f be a PCL formula. Then the w_{DM}PCL closure of f is equivalent to the PCL closure of f.

We will therefore use the same symbol $\sim$ for both operations.

Syntax and Semantics of w_{DM}PCL

Proposition (w_{DM}PCL complement $\equiv$ PCL complement)

Let f be a PCL formula, and $\neg : K \to K$ the complement function of a De Morgan Algebra $\mathcal{K}$. Then the w_{DM}PCL complementation of f is equivalent to the PCL complementation of f.

We will therefore use the same symbol $\neg$ for both operations.

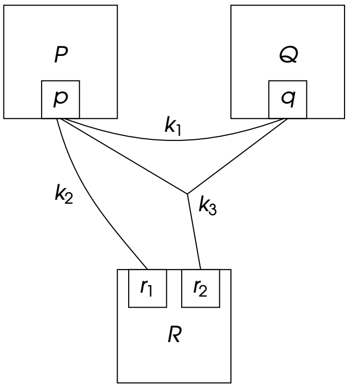
Lemma

Let f_1, f_2 be PCL formulas. Then the following w_{DM}PCL equivalences hold:

$$f_1 \uplus f_2 \equiv f_1 + f_2,$$
$$f_1 \oplus f_2 \equiv f_1 \sqcup f_2,$$
$$f_1 \otimes f_2 \equiv f_1 \wedge f_2.$$

From the results above, we can deduce that w_{DM}PCL could be similarly defined as an extension of PIL instead, without limiting its expressive power. This also implies that w_{DM}PCL could be considered a generalisation of PCL rather than an extension.

Calculation of an Architecture's weight with w_{DM}PCL



$$\gamma = \{\{p, q\}, \{p, r_1\}, \{p, q, r_2\}\}$$

PCL:

$$f = p\, q\, \overline{r_1}\, \overline{r_2} + p\, \overline{q}\, r_1\, \overline{r_2} + p\, q\, \overline{r_1}\, r_2$$
$$\gamma \models f$$

For any other configuration $\gamma' \neq \gamma$, $\gamma' \not\models f$.

w_{DM}PCL:

$$\zeta = (k_1 \otimes p\, q\, \overline{r_1}\, \overline{r_2}) \uplus (k_2 \otimes p\, \overline{q}\, r_1\, \overline{r_2}) \uplus (k_3 \otimes p\, q\, \overline{r_1}\, r_2)$$
$$\begin{aligned} \|\zeta\|(\gamma) &= (k_1 \otimes k_2 \otimes k_3) \otimes \|pq\overline{r_1}\,\overline{r_2} \uplus p\overline{q}r_1\overline{r_2} \uplus pq\overline{r_1}r_2\|(\gamma) \\ &= (k_1 \otimes k_2 \otimes k_3) \otimes \|pq\overline{r_1}\,\overline{r_2} + p\overline{q}r_1\overline{r_2} + pq\overline{r_1}r_2\|(\gamma) \\ &= (k_1 \otimes k_2 \otimes k_3) \otimes 1 \\ &= k_1 \otimes k_2 \otimes k_3 \end{aligned}$$

which is the infimum of the weights k_1, k_2 and k_3.
For any other configuration $\gamma' \neq \gamma$, $\|\zeta\|(\gamma') = 0$.

Full Normal Form of $\mathrm{w_{DM}PCL}$

Definition

A $\mathrm{w_{DM}PCL}$ formula $\zeta \in PCL(\mathcal{K}, P)$ is said to be in **full normal form** if either:

- $\zeta = k$ where $k \in K$, or
- there exist finite index sets I and J_i for every $i \in I$ such that

$$\zeta = \bigoplus_{i \in I} \left(k_i \otimes \sum_{j \in J_i} m_{i,j} \right).$$

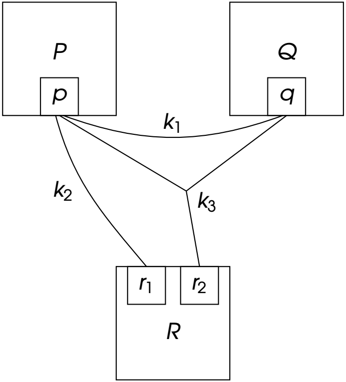
  where $k_i \in K$ and $m_{i,j}$ are full monomials.

Proposition

For any formula $\zeta \in PCL(\mathcal{K}, P)$ that is in full normal form, there exists an equivalent formula $\zeta' \in PCL(\mathcal{K}, P)$ in the full normal form $\bigoplus_{i \in I} \left(k_i \otimes \sum_{j \in J_i} m_{i,j} \right)$ that also satisfies the following statements:

1. For every $i \in I$ and $j, j' \in J_i$, $j \neq j'$ implies $m_{i,j} \not\equiv m_{i,j'}$
2. For every $i, i' \in I$, $i \neq i'$ implies $\sum_{j \in J_i} m_{i,j} \not\equiv \sum_{j \in J_{i'}} m_{i',j}$

Full Normal Form of w_{DM}PCL



We recall the formula of our example:

$$\begin{aligned}\zeta &= (k_1 \otimes p\, q\, \overline{r_1}\, \overline{r_2}) \uplus (k_2 \otimes p\, \overline{q}\, r_1\, \overline{r_2}) \uplus (k_3 \otimes p\, q\, \overline{r_1}\, r_2)\\ &\equiv (k_1 \otimes k_2 \otimes k_3) \otimes (pq\overline{r_1}\,\overline{r_2} \uplus p\overline{q}r_1\overline{r_2} \uplus pq\overline{r_1}r_2)\\ &\equiv (k_1 \otimes k_2 \otimes k_3) \otimes (pq\overline{r_1}\,\overline{r_2} + p\overline{q}r_1\overline{r_2} + pq\overline{r_1}r_2)\end{aligned}$$

which is a full normal form.

Does every formula have an equivalent full normal form?

Full Normal Form of w_{DM}PCL

Theorem

Let $\mathcal{K} = (K, \oplus, \otimes, 0, 1, \neg)$ be a De Morgan algebra and P a set of ports. Then, for every w_{DM}PCL formula $\zeta \in PCL(\mathcal{K}, P)$ we can effectively construct an equivalent formula $\zeta' \in PCL(\mathcal{K}, P)$ in full normal form which is unique up to the equivalence relation in doubly exponential time.

The calculation of an equivalent full normal form trivialises the process of computing the weight of a formula given a configuration. Since each coalescing of full monomials is satisfied (in PCL) by exactly one configuration, if the given configuration corresponds to a coalescing of the full normal form formula, then the weight of the polynomial is the respective weight attached to that coalescing (with the infimum operator), otherwise it is zero.

Decidability of Equivalence of two formulas

Theorem

Let $\mathcal{K}$ be a De Morgan Algebra and P be a set of ports. Then, for every $\zeta, \xi \in PCL(\mathcal{K}, P)$, the equality $\|\zeta\| = \|\xi\|$ is decidable in doubly exponential time.

We can prove that $\|\zeta\| = \|\xi\|$ iff for their full normal forms $\zeta' = \bigoplus_{i\in I}\left(k_i \otimes \sum_{j\in J_i} m_{i,j}\right)$ and $\xi' = \bigoplus_{l\in L}\left(k'_l \otimes \sum_{r\in M_l} m'_{l,r}\right)$ (that satisfy the requirements of that last proposition) the following 3 requirements hold:

- $\mathrm{card}(I) = \mathrm{card}(L)$, $\{k_i \mid i \in I\} = \{k'_l \mid l \in L\}$ and
- depending on the cardinality of I (and L) we require one of the following:
  - if $\mathrm{card}(I) = \mathrm{card}(\{k_i \mid i \in I\})$, then for every $i \in I, l \in L$ such that $k_i = k'_l$,
  $$\sum_{j\in J_i} m_{i,j} \equiv \sum_{r\in M_l} m'_{l,r},$$
  - if $\mathrm{card}(I) > \mathrm{card}(\{k_i \mid i \in I\})$, then
  $$\zeta' \equiv \bigoplus_{i'\in I'}\left(k_{i'} \otimes \bigsqcup_{i\in R_{i'}} \sum_{j\in J_i} m_{i,j}\right) \text{ and } \xi' \equiv \bigoplus_{l'\in L'}\left(k'_{l'} \otimes \bigsqcup_{l\in R_{l'}} \sum_{r\in M_l} m'_{l,r}\right)$$
  and, for every $i' \in I', l' \in L'$ such that $k_{i'} = k'_{l'}$,
  $$\bigsqcup_{i\in R_{i'}} \sum_{j\in J_i} m_{i,j} \equiv \bigsqcup_{l\in R_{l'}} \sum_{r\in M_l} m'_{l,r}.$$

Partial Order over polynomials of $\mathrm{w_{DM}}$PCL formulas

Definition

We define the order $\preccurlyeq$ over the polynomials of $\mathrm{w_{DM}}$PCL as follows:

$$\|\zeta_1\| \preccurlyeq \|\zeta_2\| \text{ iff } \forall \gamma \in C(P) : \|\zeta_1\|(\gamma) \leq \|\zeta_2\|(\gamma).$$

Lemma

The set of all $\mathrm{w_{DM}}$PCL polynomials over a set of ports P and a De Morgan algebra $\mathcal{K}$, $K\langle C(P)\rangle$ is a distributive lattice with the polynomial order $\preccurlyeq$, with the supremum and infimum operators being $\oplus$ and $\otimes$, respectively.

Corollary

Let $\mathcal{K} = (K, \oplus, \otimes, 0, 1, \neg)$ be a De Morgan algebra with an order $\leq$. Then, for any non-empty set of ports P, $(K\langle C(P)\rangle, \oplus, \otimes, \|0\|, \|1\|, \neg)$ is also a De Morgan algebra with the corresponding polynomial order $\preccurlyeq$.

Partial Order over polynomials of w_{DM}PCL formulas

Example

Let $P = \{p, q\}$, $\zeta = k \otimes (p \oplus pq)$ and $\xi = (k \otimes p) \oplus pq$ for some $k \in K$. Then:

Does it hold that $\|\zeta\| \preccurlyeq \|\xi\|$?

$\gamma \in C(P)$	$\|\zeta\|(\gamma)$	$\|\xi\|(\gamma)$
{{p}}	k	k
{{q}}	0	0
{{p,q}}	k	1
{{p},{q}}	0	0
{{p},{p,q}}	k	k
{{q},{p,q}}	0	0
{{p},{q},{p,q}}	0	0

Therefore, since $\forall \gamma \in C(P) : \|\zeta\|(\gamma) \leq \|\xi\|(\gamma)$ then $\|\zeta\| \preccurlyeq \|\xi\|$.

Can we always decide whether two polynomials are ordered or not?

Partial Order over polynomials of w_{DM}PCL formulas

Theorem

Let $\mathcal{K}$ be a De Morgan Algebra and P be a set of ports. Then, for every $\zeta, \xi \in PCL(\mathcal{K}, P)$, the relation $||\zeta|| \preccurlyeq ||\xi||$ is decidable in doubly exponential time.

We can prove that $||\zeta|| \preccurlyeq ||\xi||$ iff for their full normal forms

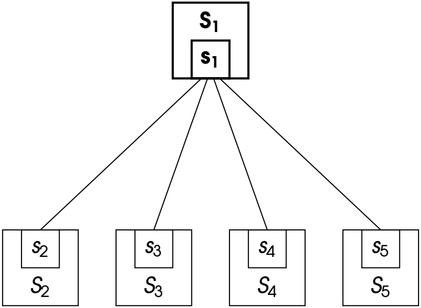
$$\zeta' = \bigoplus_{i \in I} \left(k_i \otimes \sum_{j \in J_i} m_{i,j} \right) \text{ and } \xi' = \bigoplus_{l \in L} \left(k'_l \otimes \sum_{r \in M_l} m'_{l,r} \right)$$

(that satisfy the requirements of that last proposition) the following requirement holds:

- For all $i \in I$ there exists an $l \in L$ such that:

$$\sum_{j \in J_i} m_{i,j} \equiv \sum_{r \in M_l} m'_{l,r} \text{ and } k_i \leq k'_l .$$

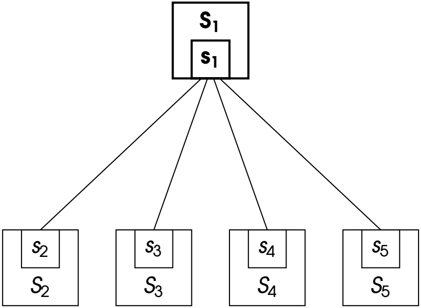
The Star architecture



The *Star Architecture* is an architecture defined for a set of components of the same type.

- One of the components, called the **central component**, is connected with all the others through a binary interaction.
- There are no interactions between non-central components

The Star architecture



If we let S_1 be the central component and:

$$f_1 := s_1 s_2 + s_1 s_3 + s_1 s_4 + s_1 s_5$$
$$r_1 := \overline{s_2 s_3} \wedge \overline{s_2 s_4} \wedge \overline{s_2 s_5} \wedge \overline{s_3 s_4} \wedge \overline{s_3 s_5} \wedge \overline{s_4 s_5}$$

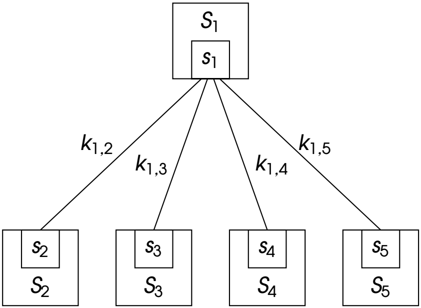
then the formula $g_1 = f_1 \wedge r_1$ is only satisfied by the displayed configuration, which is the only one that satisfies both requirements of the architecture.

If any one of the components could become the central one then we let:

$$f_i := \sum_{j \in I \setminus \{i\}} s_i s_j \text{ and } r_i := \bigwedge_{\substack{j_1, j_2 \in I \setminus \{i\}, \\ j_1 \neq j_2}} \overline{s_{j_1} s_{j_2}}$$

for $I = \{1, 2, 3, 4, 5\}$ then the formula $g = \bigsqcup_{i \in I} (f_i \wedge r_i)$ is only satisfied by such configurations.

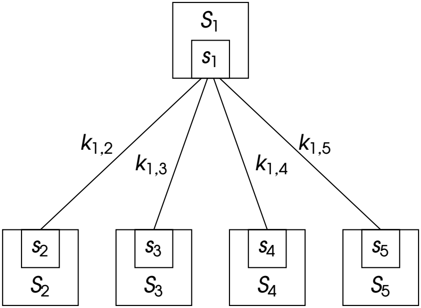
The Weighted Star architecture



In the weighted case of the Star Architecture we assign a weight to each interaction between components

- As explained previously, the weight of the architecture will be the **infimum of weights** of all interactions of the configuration's central component.
- If any interactions between non-central components are present in the configuration, then the De Morgan complement of that interaction's weight should also be included in the calculation.

The Weighted Star architecture



We take the formulas f_i, r_i of the unweighted PCL case, replace the operators with their equivalent weighted ones, and the monomials with their corresponding weighted formulas. We then end up with the formulas:

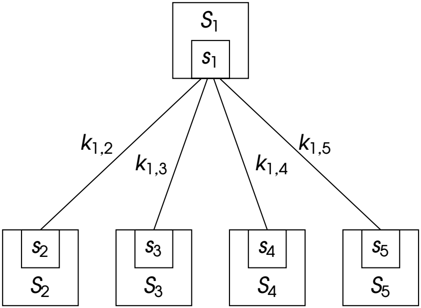
$$\zeta_i := \biguplus_{j \in I \setminus \{i\}} (k_{i,j} \otimes s_i s_j)$$

$$r_i' := \bigotimes_{\substack{j_1, j_2 \in I \setminus \{i\}, \\ j_1 \neq j_2}} (\neg k_{j_1, j_2} \oplus \overline{s_{j_1} s_{j_2}})$$

If we define $g = \bigoplus_{i \in I} (\zeta_i \otimes r_i')$, then the weight of a configuration γ is given by the value $\|g\|(\gamma)$.

If however we have several possible interactions available and we want to calculate the best obtainable weight we need to calculate $\| \sim g\|(\gamma)$.

The Weighted Star architecture

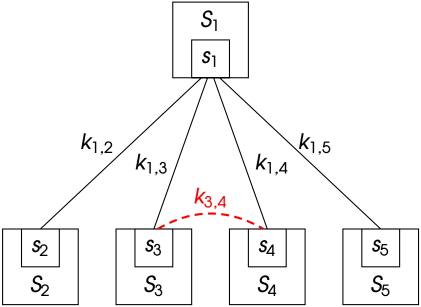


We let $\gamma = \{\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_1, s_5\}\}$.
Then we have:

$$\begin{aligned}
\| \sim g\|(\gamma) &= \left\| \sim \bigoplus_{i \in I} (\zeta_i \otimes r_i) \right\| (\gamma) \\
&= \bigoplus_{\emptyset \neq \gamma_1 \subseteq \gamma} \left\| \bigoplus_{i \in I} (\zeta_i \otimes r_i) \right\| (\gamma_1) \\
&= \|\zeta_1 \otimes r_1\| (\gamma) \\
&= \|\zeta_1\|(\gamma) \otimes \|r_1\|(\gamma) \\
&= k_{1,2} \otimes k_{1,3} \otimes k_{1,4} \otimes k_{1,5} \otimes 1 \\
&= k_{1,2} \otimes k_{1,3} \otimes k_{1,4} \otimes k_{1,5}.
\end{aligned}$$

which is the infimum of the displayed weights.

The Weighted Star architecture

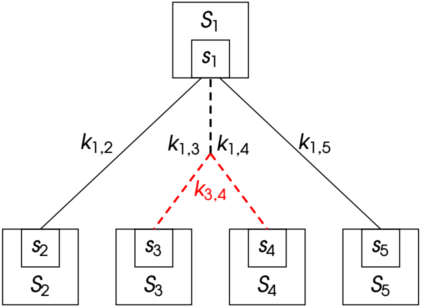


If we add an interaction between non-central components, for example we let $\gamma' = \gamma \cup \{\{s_3, s_4\}\} = \{\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_1, s_5\}, \{s_3, s_4\}\}$, then we have:

$$\begin{aligned} \| \sim g\|(\gamma') &= \left\| \sim \bigoplus_{i \in I} (\zeta_i \otimes r_i) \right\| (\gamma') \\ &= \bigoplus_{\emptyset \neq \gamma_1 \subseteq \gamma'} \left\| \bigoplus_{i \in I} (\zeta_i \otimes r_i) \right\| (\gamma_1) \\ &= \|\zeta_1 \otimes r_1\| (\gamma) \oplus \|\zeta_1 \otimes r_1\| (\gamma') \\ &= \|\zeta_1\|(\gamma) \oplus (\|\zeta_1\|(\gamma') \otimes \|r_1\|(\gamma')) \\ &= k_{1,2} \otimes k_{1,3} \otimes k_{1,4} \otimes k_{1,5} \end{aligned}$$

Notice that because $\|\zeta_1\|(\gamma') = 0$, $k_{3,4}$ is not included in the calculation. In this case, the closure operator "removed" the interaction $\{s_3, s_4\}$, therefore the result is the same as before.

The Weighted Star architecture

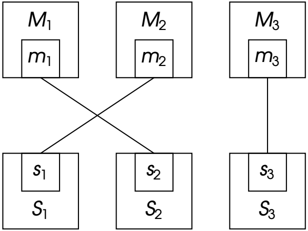


If however, we force a disallowed interaction into the final configuration of the weight and let $\gamma = \{\{s_1, s_2\}, \{s_1, s_3, s_4\}, \{s_1, s_5\}\}$, then we have:

$$\begin{aligned}
\| \sim g\|(\gamma) &= \left\| \sim \bigoplus_{i \in I} (\zeta_i \otimes r_i) \right\| (\gamma) \\
&= \bigoplus_{\emptyset \neq \gamma_1 \subseteq \gamma} \left\| \bigoplus_{i \in I} (\zeta_i \otimes r_i) \right\| (\gamma_1) \\
&= \|\zeta_1 \otimes r_1\| (\gamma) \\
&= \|\zeta_1\| (\gamma) \otimes \|r_1\| (\gamma) \\
&= k_{1,2} \otimes k_{1,3} \otimes k_{1,4} \otimes k_{1,5} \otimes \neg k_{3,4}
\end{aligned}$$

Since a complement of a weight is included in the result, then we have that the configuration cannot satisfy every qualitative requirement of the architecture.

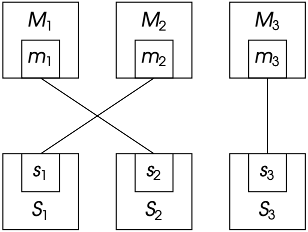
The Master/ Slave architecture



The Master/Slave architecture includes two types of components, masters and slaves.

- Every slave component must interact with exactly one master component, through their respective ports, and no other interactions can take place.

The Master/ Slave architecture



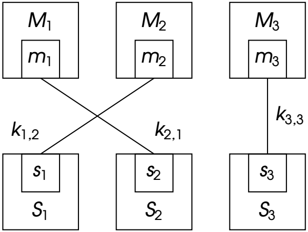
Since every slave component must interact with exactly one master component, we define the formulas:

$$\begin{aligned}
\phi_1 :&= (s_1 m_1 \wedge \overline{s_1 m_2} \wedge \overline{s_1 m_3}) \sqcup (\overline{s_1 m_1} \wedge s_1 m_2 \wedge \overline{s_1 m_3}) \\
&\sqcup (\overline{s_1 m_1} \wedge \overline{s_1 m_2} \wedge s_1 m_3) \\
\phi_2 :&= (s_2 m_1 \wedge \overline{s_2 m_2} \wedge \overline{s_2 m_3}) \sqcup (\overline{s_2 m_1} \wedge s_2 m_2 \wedge \overline{s_2 m_3}) \\
&\sqcup (\overline{s_2 m_1} \wedge \overline{s_2 m_2} \wedge s_2 m_3) \\
\phi_3 :&= (s_3 m_1 \wedge \overline{s_3 m_2} \wedge \overline{s_3 m_3}) \sqcup (\overline{s_3 m_1} \wedge s_3 m_2 \wedge \overline{s_3 m_3}) \\
&\sqcup (\overline{s_3 m_1} \wedge \overline{s_3 m_2} \wedge s_3 m_3) \\
r :&= \overline{s_1 s_2} \wedge \overline{s_1 s_3} \wedge \overline{s_2 s_3} \wedge \overline{m_1 m_2} \wedge \overline{m_1 m_3} \wedge \overline{m_2 m_3}
\end{aligned}$$

and then the corresponding PCL formula of the architecture is

$$f = (\phi_1 + \phi_2 + \phi_3) \wedge r$$

The weighted Master/ Slave architecture



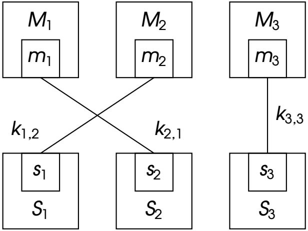
For the weighted case of the Master/Slave architecture we assume that each interaction $\{s_i, m_j\}$ with $i, j \in \{1, 2, 3\}$ has a weight $k_{i,j}$. Then we define the corresponding weighted formulas of ϕ_i as explained in the weighted Star architecture:

$$\varphi_i := \bigoplus_{j\in J}\left(k_{i,j} \otimes s_i m_j \otimes \bigotimes_{j'\in J\setminus\{j\}} (\neg k_{i,j'} \oplus \overline{s_i m_{j'}})\right),$$

where in this case $I = J = \{1, 2, 3\}$, the weight of the architecture, given a configuration, can be calculated by the polynomial of the formula:

$$\zeta =\sim (\varphi_1 \uplus \varphi_2 \uplus \varphi_3).$$

The weighted Master/ Slave architecture

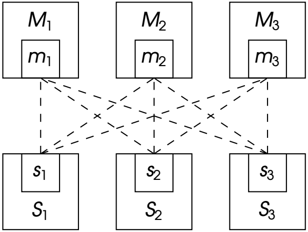


If for example we let $\gamma = \{\{s_1, m_2\}, \{s_2, m_1\}, \{s_3, m_3\}\}$ be the displayed configuration, then we have:

$$
\begin{aligned}
\|\zeta\|(\gamma) &= \| \sim (\varphi_1 \uplus \varphi_2 \uplus \varphi_3)\|(\gamma) \\
&= \bigoplus_{\gamma' \subseteq \gamma} \|\varphi_1 \uplus \varphi_2 \uplus \varphi_3\|(\gamma') \\
&= \|\varphi_1 \uplus \varphi_2 \uplus \varphi_3\|(\gamma) \\
&= \bigoplus_{\gamma_1 \cup \gamma_2 \cup \gamma_3 = \gamma} (\|\varphi_1\|(\gamma_1) \otimes \|\varphi_2\|(\gamma_2) \otimes \|\varphi_3\|(\gamma_3)) \\
&= k_{1,2} \otimes k_{2,1} \otimes k_{3,3},
\end{aligned}
$$

which is the infimum of the weights present in the interactions of γ.

The weighted Master/ Slave architecture

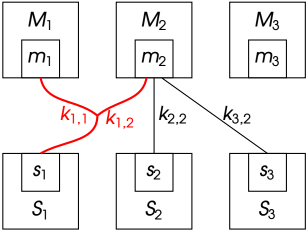


If we instead let $\gamma = \{\{s_i, m_j\} \mid i, j \in \{1, 2, 3\}\}$ be the configuration of all possible binary master/ slave interactions, then we have:

$$\begin{aligned} \|\zeta\|(\gamma) &= \bigoplus_{\gamma' \subseteq \gamma} \|\varphi_1 \uplus \varphi_2 \uplus \varphi_3\|(\gamma') \\ &= \bigoplus_{\gamma_1 \cup \gamma_2 \cup \gamma_3 \subseteq \gamma} \|\varphi_1\|(\gamma_1) \otimes \|\varphi_2\|(\gamma_2) \otimes \|\varphi_3\|(\gamma_3) \\ &= \bigoplus_{j_1, j_2, j_3 \in \{1,2,3\}} (k_{1,j_1} \otimes k_{2,j_2} \otimes k_{3,j_3}) \\ &= \left(\bigoplus_{j \in \{1,2,3\}} k_{1,j}\right) \otimes \left(\bigoplus_{j \in \{1,2,3\}} k_{2,j}\right) \otimes \left(\bigoplus_{j \in \{1,2,3\}} k_{3,j}\right), \end{aligned}$$

which is the infimum of the supremums of the weights of each slave components.

The weighted Master/ Slave architecture



We now let $\gamma = \{\{s_1, m_1, m_2\}, \{s_2, m_2\}, \{s_3, m_2\}\}$ be the displayed configuration, then we have:

$$\begin{aligned} \|\zeta\|(\gamma) &= \bigoplus_{\gamma' \subseteq \gamma} \|\varphi_1 \uplus \varphi_2 \uplus \varphi_3\|(\gamma') \\ &= \|\varphi_1 \uplus \varphi_2 \uplus \varphi_3\|(\gamma) \\ &= ((k_{1,1} \otimes \neg k_{1,2}) \oplus (\neg k_{1,1} \otimes k_{1,2})) \otimes k_{2,2} \otimes k_{3,2} \\ &= (k_{1,1} \otimes \neg k_{1,2} \otimes k_{2,2} \otimes k_{3,2}) \\ &\oplus (\neg k_{1,1} \otimes k_{1,2} \otimes k_{2,2} \otimes k_{3,2}) \end{aligned}$$

Since a complement of a weight is always included in the calculation of the architecture's weight, then there is no way to find a sub-configuration of γ that satisfies the qualitative requirements of the Master/ Slave architecture.

Conclusion and Open Problems

- w_{DM}PCL is a specification language able to describe software architectures with quantitative features.
- Its definition over De Morgan algebras allows for an interpretation of negative weights and a partial order over polynomials of formulas.
- Even with the above additional features and properties that w_{DM}PCL entails, the overall time complexity for the full normal form calculation, the decibality of equivalence and partial order procedures, is still doubly exponential.

Future work could include:

- A rewriting system for w_{DM}PCL that aims to reduce the time complexity of the equivalent full normal form calculation for cases where the doubly exponential nature of the logic can be worked around.
- An application of w_{DM}PCL's weight calculation methods for the purpose of model checking on large scale architectures.
- The investigation of first and second- order level of w_{DM}PCL for the description of architecture styles with quantitative features.

Thank you for your attention!