Bideterministic Weighted Automata

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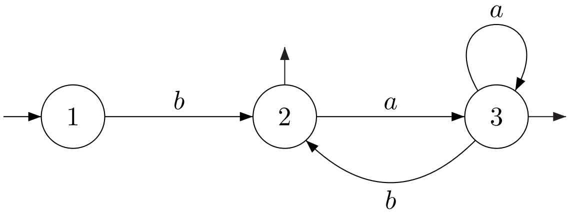

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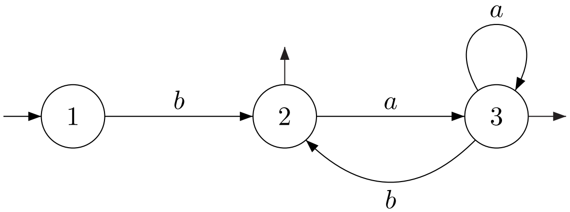

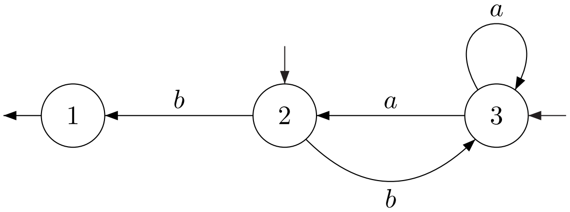

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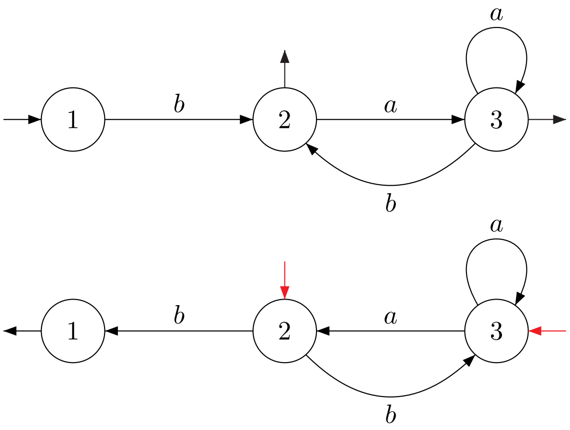

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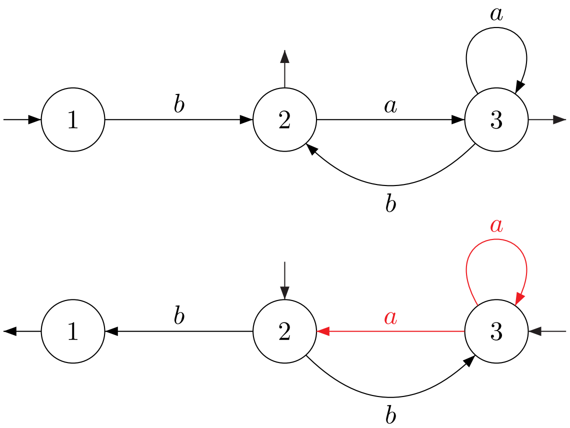

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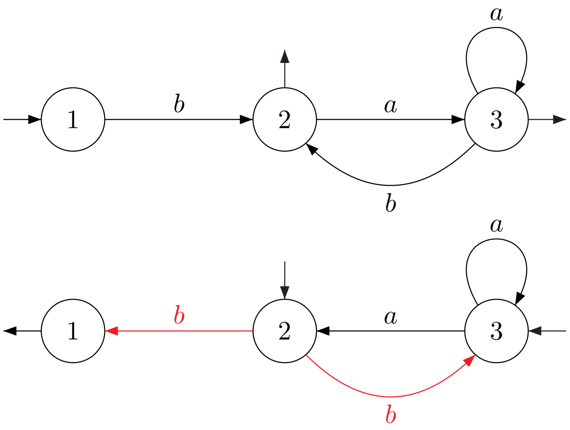

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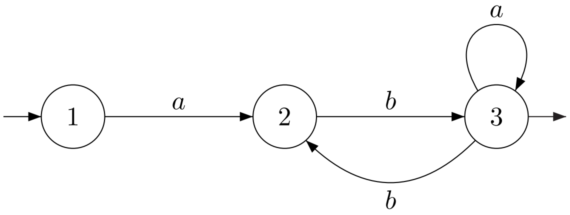

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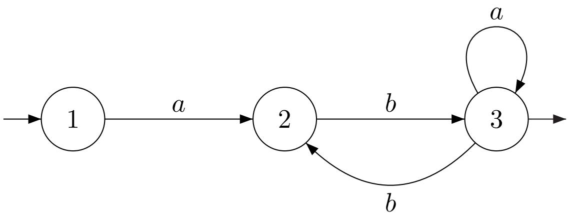

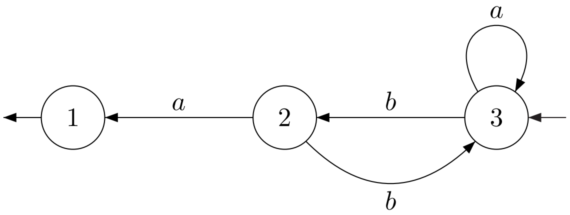

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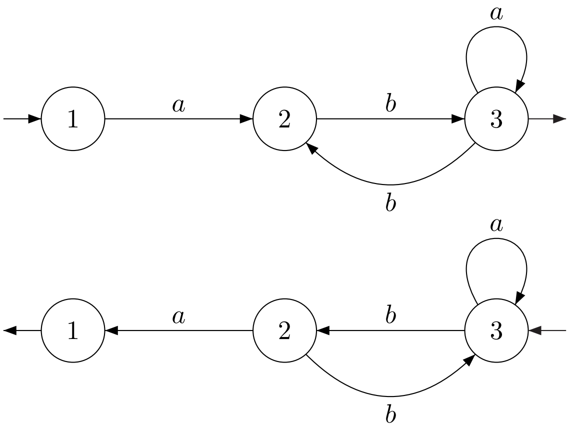

- Well-understood property, but not in a weighted setting.

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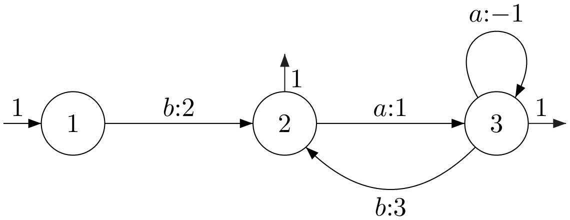

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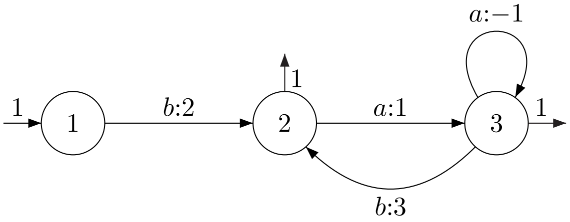

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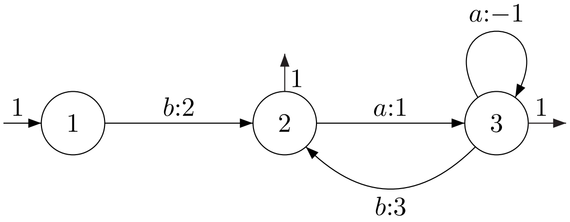

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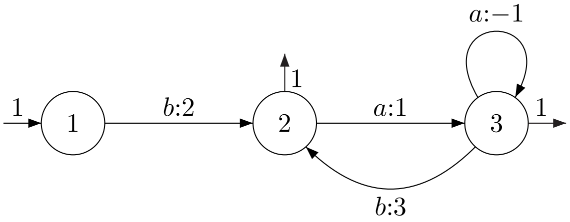

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- Value assigned to $w \in \Sigma^*$: sum of values of runs on w.

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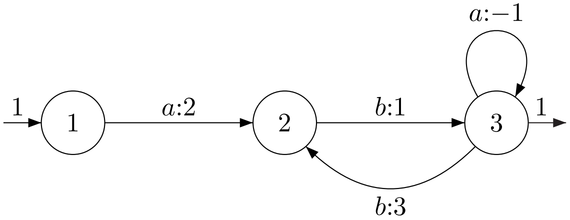

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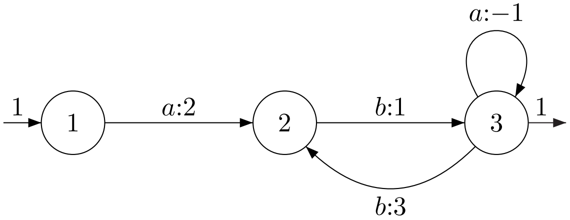

Class of weighted automata not considered so far.

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The answers depend on S. We explore some particular cases.

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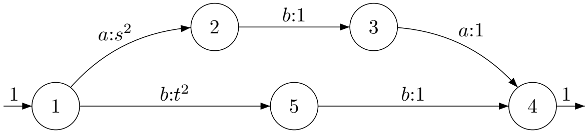

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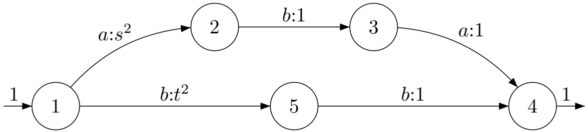

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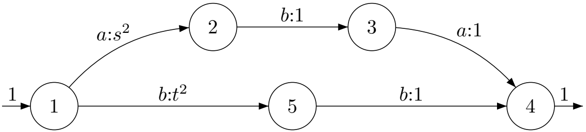

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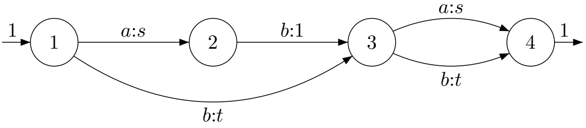

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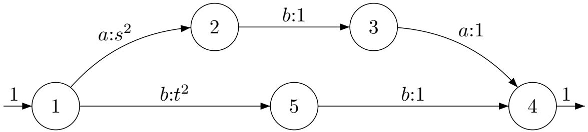

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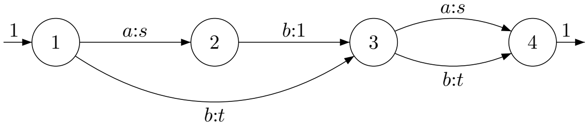

- Any bideterministic equivalent needs at least 5 states.

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- If $\mathcal{A}$ is trim bideterministic, then $\mathcal{A}'$ is as well.

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- Zero-divisor free if $ab = 0$ implies $a = 0$ or $b = 0$.
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Examples of positive semirings:

- Tropical semirings, semirings of formal languages, the Boolean semiring.

Theorem

Every trim bideterministic weighted automaton over a positive semiring S is minimal.

- Forgetting about weights of any automaton $\mathcal{A}$ over S yields an automaton $\mathcal{A}'$ recognising the support of $\|\mathcal{A}\|$.
- If $\mathcal{A}$ is trim bideterministic, then $\mathcal{A}'$ is as well.
- $\mathcal{A}'$ is minimal, so $\mathcal{A}$ has to be minimal.

Decidability of Bideterminisability: Over Fields

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Theorem

Bideterminisability of weighted automata over effective fields is decidable in polynomial time.

- Just apply the Cardon-Crochemore algorithm and find out whether the output is bideterministic.

Decidability of Bideterminisability: Over Tropical Semirings

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Bideterminisability of weighted automata over tropical semirings $\mathbb{N}_{min}$, $\mathbb{Z}_{min}$, and $\mathbb{Q}_{min}$ is decidable.

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- If $\mathcal{A}$ is bideterminisable, the minimal DFA $\mathcal{B}$ recognising the support of $\|\mathcal{A}\|$ is bideterministic.

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- If $\mathcal{B}$ is not bideterministic, $\mathcal{A}$ is not bideterminisable.
- If $\mathcal{B}$ is empty, $\mathcal{A}$ is bideterminisable.
- Otherwise $\mathcal{A}$ is bideterminisable iff we can obtain its equivalent by assigning weights to $\mathcal{B}$.

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- Can weights be assigned to a BFA $\mathcal{B}$ so that the resulting WA $\mathcal{B}'$ satisfies $\|\mathcal{B}'\| = \|\mathcal{A}\|$?

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- Let $\mathbf{x}$ be the vector of unknown weights $(x_1, \ldots, x_N)$.
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- Let $\Psi(w) = (\eta_1, \ldots, \eta_N)$.

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Decision algorithm (cont.):

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- We need $\Psi(w) \cdot \mathbf{x}^T = (\|\mathcal{A}\|, w)$ for all w recognised by $\mathcal{B}$.
- If there is a solution, the system is equivalent to a finite system for $w = w_1, \ldots, w_M$, where the $\Psi(w_i)'s$ form a basis of the vector space generated by all $\Psi(w)$.

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- It remains to check whether indeed $\|\mathcal{B}_{\mathbf{x}}\| = \|\mathcal{A}\|$.
- This can be done, as $\mathcal{B}_{\mathbf{x}}$ is deterministic.

Thank you for your attention.