Watson-Crick Powers of a Word

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Watson-Crick Powers

- Periodicity and primitivity -basic properties of words.
- Primitivity and periodicity can be generalized by replacing identity function to a suitable mapping.
- Extension to pseudo-periodicity and pseudo-primitivity [Kari et.al2010].
- Motivation from DNA computing, and the properties of information encoded as DNA strands.
- In biology, a DNA strand and its Watson-Crick complement basically contain the same information.

Watson-Crick Powers

- DNA strands : Word over alphabet $\{A, G, C, T\}$
- Watson-Crick complementarity: Antimorphic involution θ: $(\theta(uv) = \theta(v)\theta(u), \theta(\theta(a)) = a)$
- $\theta: \ A \mapsto T, \ C \mapsto G$ and vice-versa.
- The word $w = ACTG \ CAGT \ CAGT$ is repetitive (θ-periodic-not θ(pseudo)-primitive) as $w = u\theta(u)\theta(u)$ for $u = ACTG$.
- The word $ACTA$ is θ-primitive.
- Define a new binary word operation *strong θ-catenation.*
- *θ- antimorphic involution.*

Existing Generalizations

1. [Kari et.al2010] A word w was called a θ- power or pseudo-power of u if $w \in u\{u, \theta(u)\}^*$ for some $u \in \Sigma^+$, and θ-primitive or pseudo-primitive if it was not a pseudo-power of any such word.
2. [Kari and Kulkarni2014] The θ-catenation operation $\odot$ defined by
$$u \odot v = \{uv, u\theta(v)\}$$
$$u^{\odot(n)} = u\{x_1 x_2 \cdots x_{n-1} \ : \ x_i = u \text{ or } x_i = \theta(u)\}$$
3. (2) defines a binary operation that generates θ-powers defined in (1).
4. Operation $\odot$ does not generate powers of u that begins with $\theta(u)$.

Strong θ-catenation

We propose the following.

1. Define the binary operation denoted by $\otimes$ as
$$u \otimes v = \{uv, u\theta(v), \theta(u)v, \theta(u)\theta(v)\}$$
2. We call it the *strong θ-catenation* operation.
3. $u^{\otimes(n)} = \{u_1 u_2 \cdots u_n \ : \ u_i \in \{u, \theta(u)\}\}$
4. For $u = ATC$, $v = GCTA$,
$u \otimes v = \{ATC\,GCTA, ATC\,TAGC, GAT\,GCTA, GAT\,TAGC\}$.
5. $\otimes$ generates all θ-powers of u. (We call them *strong θ-powers*).
6. When θ is an antimorphic involution that formalizes the Watson-Crick complementarity, we call these strong θ-powers as *Watson-Crick powers*.

Property

Lemma

For $u \in \Sigma^+$, and ϕ (anti)morphic involution, the following statements hold.

1. *For all $n \geq 1$, we have that $\alpha \in \{u, \phi(u)\}^n$ iff $\alpha \in u^{\otimes(n)}$.*
2. *For all $n \geq 1$, we have that $u^{\otimes(n)} = \phi(u^{\otimes(n)}) = \phi(u)^{\otimes(n)}$.*
3. *For all $m, n \geq 1$, we have that*

$$(u^{\otimes(m)})^{\otimes(n)} = (u^{\otimes(n)})^{\otimes(m)} = u^{\otimes(mn)}$$

Conjugacy and Commutativity

1. For $x, y \in \Sigma^+$, x is a conjugate of y if $xz = zy$ for some $z \in \Sigma^*$.
2. If $x = aba$ is a conjugate of $y = baa$, since for $a \in \Sigma^*$,

$$aba(a) = (a)baa$$

3. For $x, y \in \Sigma^+$, x commutes with y if $xy = yx$.

Generalization to the notion of pseudo-conjugacy and pseudo-commutativity

[Kari and Mahalingam2005] θ-conjugacy and θ-commutativity.

1. For $x, y \in \Sigma^+$, x is a θ-conjugate of y if $xz = \theta(z)y$ for some $z \in \Sigma^*$.
2. For $x, y \in \Sigma^+$, x θ-commutes with y if $xy = \theta(y)x$.
3. Let θ be an antimorphic involution, such that $\theta(a) = b, \theta(b) = a$ for $a, b \in \Sigma$.
4. Then, $x = aba$ is a θ-conjugate of $y = aab$, since for $z = ab = \theta(z)$
$$aba(ab) = \theta(ab)aab$$
5. $x = abbaab$ θ-commutes with $y = baab$ since
$$abbaab(baab) = \theta(baab)abbaab$$

$\otimes$-conjugacy and $\otimes$-commutativity

1. For $u, w \in \Sigma^+$, such that u is a $\otimes$- conjugate of w, if

$$u \otimes v = v \otimes w$$

for some $v \in \Sigma^+$.

2. For $u, v \in \Sigma^+$, u $\otimes$-commutes with v if

$$u \otimes v = v \otimes u$$

Characterization

$\otimes$-conjugacy

Theorem

Let $u, v, w \in \Sigma^+$. *Then,* $u \otimes v = v \otimes w$ *iff one of the following holds:*

- $u = \theta(v) = w$
- $u = v = w$
- $u = \theta(v) = \theta(w)$
- $u = v = \theta(w)$
- $u = s^m = w$ *and* $v = s^n$*, for* $s \in \mathrm{P}_\theta$.

where P_θ *denotes the set of all* θ*-palindromes.*

Characterization

$\otimes$-commutativity

Theorem

Let $u, v \in \Sigma^+$. *Then,* $u \otimes v = v \otimes u$ *iff one of the following holds:*

- $u = \theta(v)$
- $u = v$
- $u = s^m$ *and* $v = s^n$, *for* $s \in P_\theta$.

$\circ$-Primitivity

1. A nonempty word w is called $\circ$-primitive if $w \in u^{\circ(i)}$ for some word u and $i \geq 1$ yields $i = 1$ and $w = u$.
2. [Kari et.al2010] A word w is called θ-primitive if it cannot be expressed as a non-trivial θ-power (pseudo-power) of another word.
3. [Kari and Kulkarni2014] A nonempty word w is called $\odot$-primitive if $w \in u^{\odot(i)}$ for some word u and $i \geq 1$ yields $i = 1$ and $w = u$.
4. A nonempty word w is called $\otimes$-primitive if $w \in u^{\otimes(i)}$ for some word u and $i \geq 1$ yields $i = 1$ and $w = u$ or $w = \theta(u)$.

Example

1. Consider the Watson-Crick complementarity function θ_{DNA} : θ maps $A \mapsto T$, $C \mapsto G$ and vice-versa.
2. Let $w = ACTAGTAGTACTACTAGT$. The word w is not $\otimes$-primitive with respect to θ_{DNA} since $w \in (ACT)^{\otimes(6)}$.
3. The word $x = ACTAAG$ is $\otimes$-primitive with respect to θ_{DNA}.

Relation between existing (pseudo)-primitive words

Given an antimorphic involution θ and a word u in Σ^+, the following are equivalent:

1. u is θ-primitive,
2. u is $\otimes$-primitive with respect to θ
3. [Kari and Kulkarni2014] u is $\odot$-primitive with respect to θ.

Note that all $\otimes$-primitive words with respect to a given antimorphic involution θ are primitive, but the converse does not generally hold.

Primitive power and primitive root

Lemma

Let θ be an antimorphic involution on Σ^*. For all $w \in \Sigma^+$, there exists a word u which is $\otimes$-primitive with respect to θ, such that $w \in u^{\otimes(n)}$ for some $n \geq 1$.

Definition

For a binary operation $\circ$, a $\circ$-primitive word u is called a "$\circ$-root of w," if $w \in u^{\circ(n)}$ for some $n \geq 1$.

Note that a word w may have several $\circ$-roots.

Example

1. For the word $w = ACT\,AGTAGT\,ACTACT\,AGT$, we have that $w \in x^{\otimes(6)} = (ACT)^{\otimes(6)} = (\theta(x))^{\otimes(6)} = (AGT)^{\otimes(6)}$
2. There are two $\otimes$-primitive words, x and $\theta(x)$, which are $\otimes$-roots of w.
3. However, uniqueness can still be ensured if we select the θ-pair $x_\theta = \{x, \theta(x)\}$, such that x is $\otimes$-primitive and $w \in x_\theta^+$.

Primitive root pair

Definition

Given an antimorphic involution θ, the $\otimes$-primitive root pair of a word $w \in \Sigma^+$ relative to θ (or simply the $\otimes$-primitive root pair of w) is the θ-pair $u_\theta = \{u, \theta(u)\}$ which satisfies the property that u is $\otimes$-primitive and $w \in u^{\otimes(n)}$ for some $n \geq 1$.

1. The $\otimes$-primitive root pair of w is $x_\theta = \{ACT, AGT\}$.
2. We denote the $\otimes$-primitive root pair relative to θ, by $\rho_\theta^{\otimes}(w)$.

Existence and uniqueness of primitive root

Theorem

[Kari et.al2010] Let $u, v, w \in \Sigma^+$ such that $w \in u\{u, \theta(u)\}^* \cap v\{v, \theta(v)\}^*$. Then u and v have a common θ-primitive root.

Theorem

Given an antimorphic involution θ and a word $w \in \Sigma^+$, its $\otimes$-primitive root pair $\rho_\theta^\otimes(w)$ always exists and is unique.

Conclusions and Future Work

1. Defined a new binary operation $\otimes$.
2. Defined strong θ-powers based on $\otimes$.
3. Studied some combinatorial properties based on $\otimes$.
4. Possible extension to the study of languages with respect to this operation.
5. Study of extension of bicatenation operation defined by $u \star v = \{uv, vu\}$ with respect to $\otimes$.