Some Algebraic Ways to Calculate Zeros of the Riemann Zeta Function

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Plan of the talk

1. Number-theoretical prerequisites

2. The Riemann Hypothesis and complexity theory

3. Some algebraic ways to calculate zeros of the Riemann zeta function

Part 1. Number-theoretical prerequisites

The Riemann zeta function – powerful tool for studying prime numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots$$

The Riemann zeta function can be defined by a *Dirichlet series*, namely,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Here s is a complex number

The series converges for $\mathrm{Re}(s) > 1$ but the zeta function can be analytically extended to the whole complex plane with the exception of point $s = 1$ because the harmonic series diverges:

$$\zeta(1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \infty$$

Where are the prime numbers?

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots \qquad \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Leonhard Euler:

$$\begin{aligned}
\zeta(s) &= \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}} \\
&= \prod_{p \text{ is prime}} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \ldots\right) \\
&= \sum_{p_1, \ldots, p_k \text{ are primes}} \left(p_1^{m_1} \ldots p_k^{m_k}\right)^{-s} \\
&= \sum_{n=1}^{\infty} n^{-s}
\end{aligned}$$

Fundamental Theorem of Arithmetic. *Every natural number can be represented as a product of powers of prime numbers, and in a unique way.*

The infinitude of prime numbers

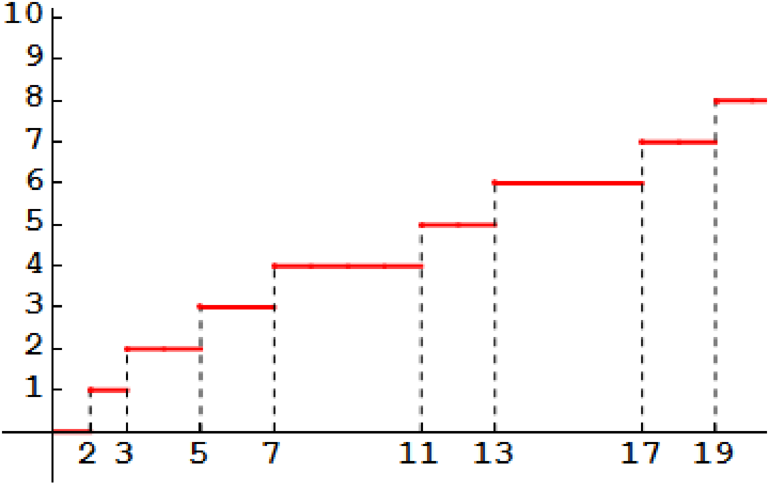
Theorem (Euclid, 3rd century B.C.) *There are infinitely many prime numbers.*

Proof (Euler, 18th century A.D.) *If the number of primes would be finite, then the (divergent) harmonic series would have finite value:*

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \zeta(1) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p}}$$

How many primes are less or equal to a given bound x?

$$\pi(x) = \text{the number of primes } p \text{ such that } p \leq x$$



Guess of Gauss

Conjecture (Karl Gauss)

$$\pi(x) \approx \int_2^x \frac{\mathrm{d}y}{\ln(y)} = \mathrm{Li}(x) \approx \frac{x}{\ln(x)}$$

$\mathrm{Li}(x)$ is known as the *logarithmic integral function*

Informally speaking, one can say that in the vicinity of y the "probability" of an integer to be a prime number is equal to $1/\ln(y)$.

Explicit formula for $\pi(x)$ given by Riemann

$$\pi(x) = J(x) + \frac{-1}{2} J(x^{1/2}) + \cdots + \frac{\mu(n)}{n} J(x^{1/n}) + \ldots$$

$\mu(n)$ is the Möbius function:

$$\mu(n) = \begin{cases} (-1)^m, & \text{if } n \text{ is a product of } m \text{ different primes} \\ 0, & \text{otherwise} \end{cases}$$

$$J(x) = \mathrm{Li}(x) - \sum_{\zeta(\rho)=0} \mathrm{Li}(x^\rho) - \ln(2) + \int_x^\infty \frac{\mathrm{d}t}{t(t^2-1)\ln(t)}$$

$$\pi(x) = \mathrm{Li}(x) + \ldots$$

Zeros of the zeta function

Euler: $0 = \zeta(-2) = \zeta(-4) = \cdots = \zeta(-2m) = \ldots$

Negative even integer called the *trivial zeros* of the zeta function

Riemann (1859): *All other (called non-trivial zeros) of the zeta function are non-real and satisfy*

$$0 \leq \mathrm{Re}(s) \leq 1$$

J. Hadamard and Ch. de la Vallée Poussin (1896, idependently): *All the non-trivial zeta zeros satisfy*

$$0 < \mathrm{Re}(s) < 1$$

Corollary (Prime Number Theorem)

$$\pi(x) = \mathrm{Li}(x) + o(x/\ln(x))$$

The Riemann Hypothesis

$\mathbf{RH}$: *All the non-trivial zeta zeros satisfy*

$$\mathrm{Re}(s) = \frac{1}{2}$$

Corollary: $\mathbf{RH} \Rightarrow \pi(x) = \mathrm{Li}(x) + O(x^{1/2}\ln(x))$

Equivalent formulation: $\mathbf{RH} \Leftrightarrow \pi(x) = \mathrm{Li}(x) + O(x^{1/2}\ln(x))$

Part 2. The Riemann Hypothesis and complexity theory

Testing the primality

M. Agrawal, N. Kayal and N. Saxena [2002]: *The primality of a number p can be recognized in polynomial time*

Original bound: $O(\log^{12+\epsilon}(p))$

H. W. Lenstra and C. Pomerance [2005]: $O(\log^{6+\epsilon}(p))$

G. L. Miller [1976]: *The primality of a number p can be recognized in time complexity*

$$O(\ln^{4+\epsilon}(p))$$

under the assumption of the validity of (generalized) Riemann Hypothesis

Alan Turing and the Riemann Hypothesis

1. A. Turing invented an analogous machine for calculation of the values of the zeta function. In 1939 he got a grant for its physical implementation and started the construction. The process was broken by the WW2 and never resumed.

2. After the war Turing was the first who used a digital computer for verifying the Riemann Hypothesis. To this end he substantially improved the known technique. So called "Turing's method" is in use up to now for verifying the hypothesis for the initial zeros of the zeta function.

3. In his logical dissertation [1939], among other things, Turing estimates the complexity of the *statement* of the Riemann Hypothesis.

The Riemann Hypothesis in the arithmetical hierarchy

Kurt Gödel: A very powerful general technique of *arithmetization*

Equivalent formulation of $\mathbf{RH}$:

$$\mathbf{RH} \Leftrightarrow \pi(x) = \mathrm{Li}(x) + O(x^{1/2}\ln(x))$$

Turing [1939]: $\mathbf{RH}$ is in Π^0_2

$$\mathbf{RH} \Leftrightarrow \forall x_1 \dots x_n \exists y_1 \dots y_m [T(x_1, \dots, x_n, y_1, \dots, y_m)]$$

Georg Kreisel [1958]: $\mathbf{RH}$ is in Π^0_1

$$\mathbf{RH} \Leftrightarrow \forall x_1 \dots x_n [K(x_1, \dots, x_n)]$$

Hilbert's 10th problem

DPRM-theorem (M. Davis, H. Putnam, J. Robinson, and Yu. Matiyasevich [1970]) *Every formula from* Π_1^0 *with parameters* $a_1, \ldots, a_m$ *is equivalent to a formula of the special form*

$$\forall x_1 \ldots x_n [P(a_1, \ldots, a_m, x_1 \ldots x_n) \neq 0]$$

where P *is a polynomial with integer coefficients.*

Corollary (together with Kreisel's result): *One can construct a particular polynomial* $R(x_1 \ldots x_n)$ *with integer coefficients such that the Riemann Hypothesis is equivalent to the statement that Diophantine equation*

$$R(x_1 \ldots x_n) = 0$$

has no solutions.

Complexity inside Π^0_1

Georg Kreisel [1958]: $\mathbf{RH}$ is in Π^0_1

$$\mathbf{RH} \Leftrightarrow \forall x_1 \dots x_n [K(x_1, \dots, x_n)]$$

Cristian Calude, Elena Calude, and Michael Dinneen [2006] suggested that the complexity of a Π^0_1 statement can be defined as some measure of the simplest machine (or program) that never halts if and only if the statement is true.

Among other famous mathematical problems they estimated (from above) the complexity of the Riemann Hypothesis.

Register machines

In a series of papers C. Calude, E. Calude, and Dinneen used a version of *register machines* for estimating the complexity of mathematical problems.

Such models of computational devices were proposed in 1961 by J. Lambek, by Z. A. Melzak, and by M. L. Minsky (independently).

C. Calude and E. Calude [2006]: The Riemann Hypothesis was presented by a register machine with 290 rather powerful instructions

E. Calude [2012]: Improved to 178 instructions

Matiyasevich [2020]: register machine with 29 registers and 130 simple instructions

Turing machines

Adam Yedidia and Scott Aaronson [2016] constructed a classical Turing machine with two-letter tape alphabet which, having started with the empty tape, will never halt if and only if the Riemann Hypothesis is true.

The original machine had 5372 states

Later this was improved to 744 states

Python 3 program that never halts if and only the Riemann Hypothesis is true (Matiyasevich [2020])

```
from math import gcd
d=m=p=0
f=g=h=n=q=1
while p**2*(m-f)<h:
  d=2*n*d-(-1)**n*g
  n=n+1
  c=gcd(n,q)
  q=n*q//c
  if c==1: p=p+1
  m=0; r=q
  while r>1:
    r=r//2; m=m+d
  g=2*f
  f=2*n*f
  h=(2*n+3)*h
```

Part 3. Some algebraic ways to calculate zeros of the Riemann zeta function

Zeros of the zeta function

The zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Transcendental equation:

$$\zeta(s) = 0$$

An initial aproximation a:

$$a \approx \rho, \quad \zeta(\rho) = 0$$

Method 1. Newton's iterations

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \zeta(\rho) = 0 \qquad a \approx \rho$$

$$\begin{aligned} a_0 &= a \\ a_{n+1} &= a_n - \frac{\zeta(a_n)}{\zeta'(a_n)} \end{aligned}$$

$$a_n \underset{n \to \infty}{\longrightarrow} \rho$$

Method 2. Taylor expansion

$$q_{a,n} = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\zeta(s)\bigg|_{s=a}$$

$$P_{a,N}(s) \;=\; \sum_{n=0}^{N} \frac{q_{a,n}}{n!}(s-a)^n$$

$$\zeta(s) = P_{a,N}(s) + O\big((s-a)^{N+1}\big)$$

$$P_{a,N}(s) \approx \zeta(s)$$

$$P_{a,N}(\rho) \approx \zeta(\rho) = 0$$

$$\zeta(s) = 0 \qquad\qquad P_{a,N}(s) = 0$$

My wild idea

$$\zeta(s) = 0 \qquad\qquad P_{a,N}(s) = 0$$

$$P_{a,N}(s) = \sum_{n=0}^{N} p_{a,N,n} s^n$$

Let us replace $s^1,\ s^2,\ \ldots,\ s^N$ by independent unknowns $s_1,\ s_2,\ \ldots,\ s_N$:

$$P_{a,N}(s_1, \ldots, s_N) = p_{a,N,0} + \sum_{n=1}^{N} p_{a,N,n} s_n$$

$$P_{a,N}(s) = 0 \qquad\qquad P_{a,N}(s_1, \ldots, s_N) = 0$$

We *wish* that

$$s_2 = s_1^2,\ s_3 = s_1^3,\ \ldots,\ s_N = s_1^N$$

Where could we get more equations?

$$m^{-s}\zeta(s) = 0$$

$$m^{-s}\zeta(s) = P_{a,N,m}(s) + O\big((s-a)^{N+1}\big)$$

$$m^{-s}\zeta(s) = 0 \qquad\qquad P_{a,N,m}(s) = 0$$

$$P_{a,N,m}(s) \;=\; p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s^n$$

$$P_{a,N,m}(s_1,\ldots,s_N) \;=\; p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s_n$$

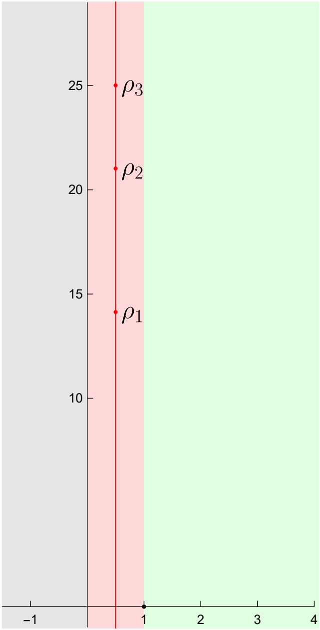
$$m^{-s}\zeta(s) = 0 \qquad\qquad P_{a,N,m}(s_1,\ldots,s_N) = 0$$

Method 3. Linear systems

$$m^{-s}\zeta(s) = 0, \qquad m = 1, \ldots N$$

$$P_{a,N,m}(s_1, \ldots, s_N) = 0, \qquad m = 1, \ldots N$$

Initial zeros of the zeta function



$$\rho_3 = 0.5 + 25.01085758014568876321 3...\mathrm{i}$$

$$\rho_2 = 0.5 + 21.02203963877155499262 8...\mathrm{i}$$

$$\rho_1 = 0.5 + 14.13472514173469379045 7...\mathrm{i}$$

Numerical example 1

$$\zeta(\rho_1) = 0 \qquad \rho_1 = 0.5 + 14.134725141734\ldots\mathrm{i}$$

$$a = 0.4 + 14\mathrm{i} \qquad N = 5$$

$$s_1 = 0.499999828490\ldots + 14.134725265432\ldots\mathrm{i}$$

$$\rho_1/s_1 = 1 + (-8.311... - 1.242...\mathrm{i}) \times 10^{-7}$$

$$\begin{aligned}
s_2/s_1^2 &= 1 + (6.228... - 4.386...\mathrm{i}) \times 10^{-9} \\
s_3/s_1^3 &= 1 + (1.733... - 1.486...\mathrm{i}) \times 10^{-8} \\
s_4/s_1^4 &= 1 + (3.171... - 3.286...\mathrm{i}) \times 10^{-8} \\
s_5/s_1^5 &= 1 + (4.749... - 5.954...\mathrm{i}) \times 10^{-8}
\end{aligned}$$

Numerical example 2

$$\zeta(\rho_2) = 0 \qquad \rho_2 = 0.5 + 21.0220396387\ldots\mathrm{i}$$

$$a = 0.4 + 21\mathrm{i} \qquad N = 5$$

$$s_1 = 0.499999989006\ldots + \ 21.022039625249\ldots\mathrm{i}$$

$$\rho_2/s_1 = 1 + (6.553... - 5.073...\mathrm{i}) \times 10^{-10}$$

$$\begin{aligned}
s_2/s_1^2 &= 1 + (1.301... + 2.431...\mathrm{i}) \times 10^{-10} \\
s_3/s_1^3 &= 1 + (4.351... + 7.115...\mathrm{i}) \times 10^{-10} \\
s_4/s_1^4 &= 1 + (.958... + 1.382...\mathrm{i}) \times 10^{-9} \\
s_5/s_1^5 &= 1 + (1.741... + 2.228...\mathrm{i}) \times 10^{-9}
\end{aligned}$$

More numerical examples for $a = 0.4 + 14\mathrm{i}$

$$N = 50: \qquad \rho_1/s_1 = 1 + (-2.549641... + 4.473122...\mathrm{i}) \times 10^{-84}$$

$$N = 75: \qquad \rho_1/s_1 = 1 + (-0.789141... - 2.357178...) \times 10^{-124}$$

$$N = 100: \qquad \rho_1/s_1 = 1 + (-6.635292... - 0.802832...) \times 10^{-165}$$

$$N = 150: \qquad \rho_1/s_1 = 1 + (\ 5.089922... - 0.495039...) \times 10^{-246}$$

Method 4. Approximations by finite Dirichlet series

$$m^{-s}\zeta(s) = P_{a,N,m}(s) + O\big((s-a)^{N+1}\big)$$

$$P_{a,N,m}(s) = p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s^n$$

$$P_{a,N,m}(s_1,\ldots,s_N) = p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s_n$$

$$P_{a,N,m}(s_1,\ldots,s_N) = 0, \quad m = 1,\ldots N$$

$$m^{-s}\zeta(s) = D_{a,N,m}(s) + O\big((s-a)^{N+1}\big)$$

$$D_{a,N,m}(s) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} n^{-s}$$

$$D_{a,N,m}(s_2,\ldots,s_N) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} n^{-s_n}$$

$$D_{a,N,m}(s_2,\ldots,s_N) = 0, \quad m = 1,\ldots N-1$$

Method 4. Approximations by finite Dirichlet series

$$P_{a,N,m}(s_1,\dots,s_N) = p_{a,N,0,m} + \sum_{n=1}^{N} p_{a,N,n,m} s_n$$

$$P_{a,N,m}(s_1,\dots,s_N) = 0, \quad m = 1,\dots N$$

$$D_{a,N,m}(s_2,\dots,s_N) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} n^{-s_n}$$

$$D_{a,N,m}(s_2,\dots,s_N) = 0, \quad m = 1,\dots N-1$$

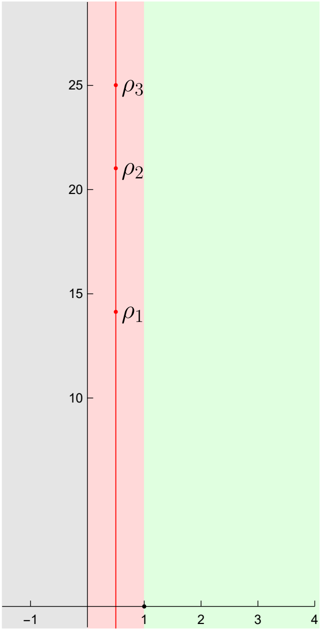
n^{-s_n} is replaced by q_n:

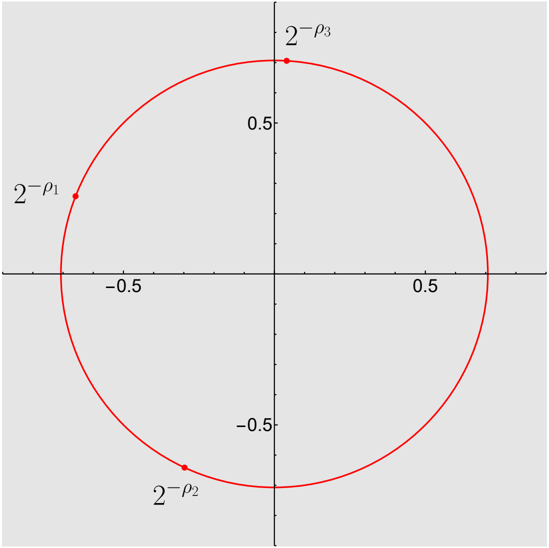
$$Q_{a,N,m}(q_2,\dots,q_N) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} q_n$$

$$Q_{a,N,m}(q_2,\dots,q_N) = 0, \quad m = 1,\dots N-1$$

Images of the zeta zeros

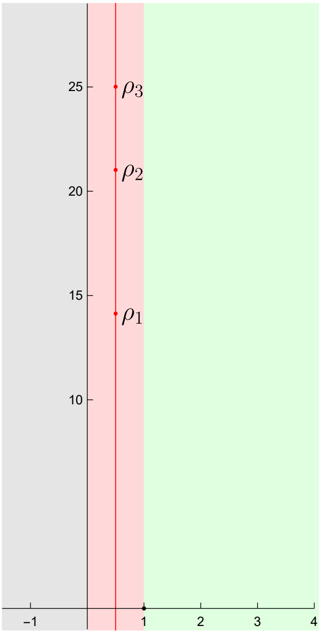
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad s \mapsto 2^{-s}$$

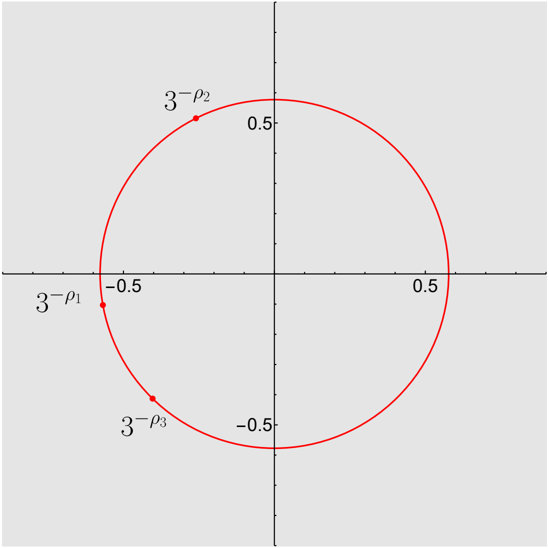




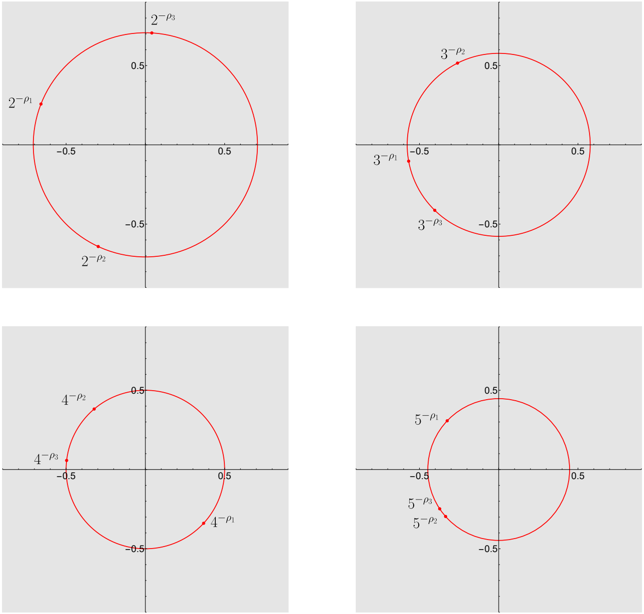
Images of the zeta zeros

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad s \mapsto 3^{-s}$$





Images of the zeta zeros



Method 4. Approximations by finite Dirichlet series

$$P_{a,N,m}(s_1,\ldots,s_N)=0,\quad m=1,\ldots N$$

We got: $s_n \approx \rho^n$

$\mathbf{RH}:\ \mathrm{Re}(s_1)\approx 1/2$

$$Q_{a,N,m}(q_2,\ldots,q_N)=0,\quad m=1,\ldots N-1$$

We expect: $q_n \approx n^{-\rho}$

$\mathbf{RH}:\ |\, q_n \,| \approx n^{-1/2}$

Numerical examples: $N = 5$

$a = 0.4 + 14\mathrm{i}$ $\zeta(\rho_1) = 0$ $\rho_1 = 0.5 + 14.134725141734\ldots\mathrm{i}$

n	q_n	$\lvert q_n/n^{-\rho_1} - 1\rvert$
2	$-0.658570722632\ldots + 0.257458025275\ldots\mathrm{i}$	$4.8927\ldots \cdot 10^{-8}$
3	$-0.568086335195\ldots - 0.103010905955\ldots\mathrm{i}$	$4.6606\ldots \cdot 10^{-8}$
4	$0.367430765659\ldots - 0.339108615925\ldots\mathrm{i}$	$5.8663\ldots \cdot 10^{-8}$
5	$-0.324829272639\ldots + 0.307385716024\ldots\mathrm{i}$	$9.3121\ldots \cdot 10^{-8}$

$a = 0.4 + 21\mathrm{i}$ $\zeta(\rho_2) = 0$ $\rho_2 = 0.5 + 21.0220396387715\ldots\mathrm{i}$

n	q_n	$\lvert q_n/n^{-\rho_2} - 1\rvert$
2	$-0.297469436794\ldots - 0.641491987361\ldots\mathrm{i}$	$3.9138\ldots \cdot 10^{-8}$
3	$-0.259863535982\ldots + 0.515562100991\ldots\mathrm{i}$	$3.5189\ldots \cdot 10^{-8}$
4	$-0.323023887992\ldots + 0.381648489607\ldots\mathrm{i}$	$1.7298\ldots \cdot 10^{-8}$
5	$-0.335079474007\ldots - 0.296178578044\ldots\mathrm{i}$	$9.9846\ldots \cdot 10^{-9}$

More numerical examples for $a = 0.4 + 14\mathrm{i}$

$$N = 100: \qquad \max_{1 \le n \le N} \left| \frac{q_n}{n^{-\rho_1}} - 1 \right| = 1.150... \times 10^{-162}$$

$$N = 150: \qquad \max_{1 \le n \le N} \left| \frac{q_n}{n^{-\rho_1}} - 1 \right| = 8.797... \times 10^{-244}$$

$$N = 200: \qquad \max_{1 \le n \le N} \left| \frac{q_n}{n^{-\rho_1}} - 1 \right| = 6.939... \times 10^{-325}$$

Method 4 (repeated)

$$m^{-s}\zeta(s) = 0$$

$$m^{-s}\zeta(s) = D_{a,N,m}(s) + O\big((s-a)^{N+1}\big)$$

$$D_{a,N,m}(s) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} n^{-s} \approx m^{-s}\zeta(s)$$

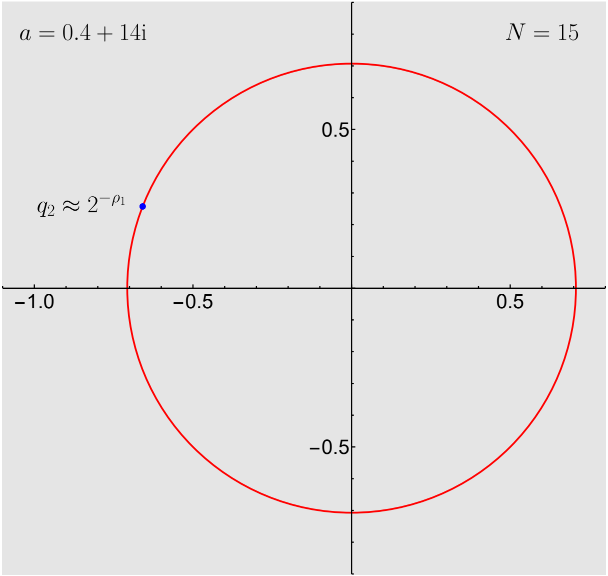
$$Q_{a,N,m}(q_2, \dots, q_N) = d_{a,N,1,m} + \sum_{n=2}^{N} d_{a,N,n,m} q_n$$

$$Q_{a,N,m}(2^{-s}, \dots, N^{-s}) \approx 0$$

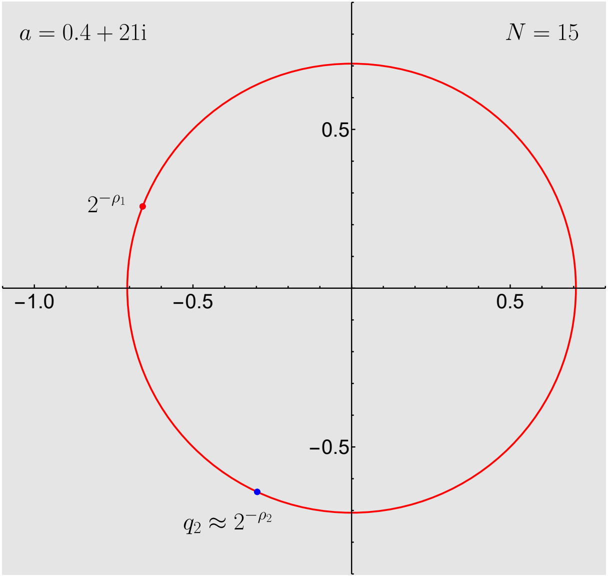
$$Q_{a,N,m}(q_2, \dots, q_N) = 0, \quad m = 1, \dots N-1$$

$$q_2 \approx 2^{-s}, \dots, q_N \approx N^{-s}$$

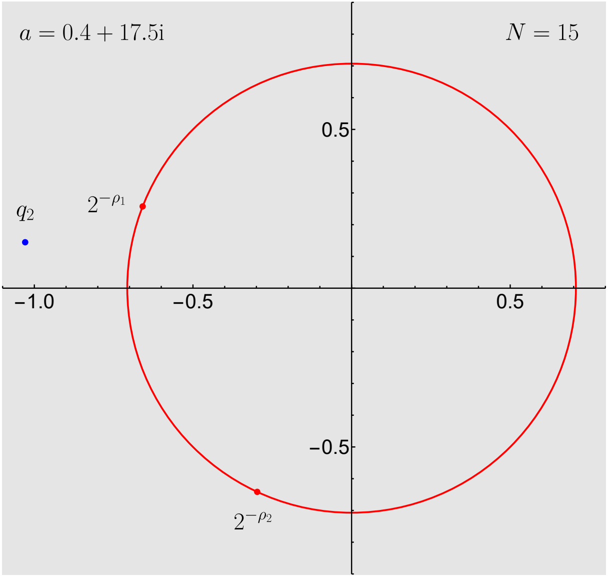
Between two zeros



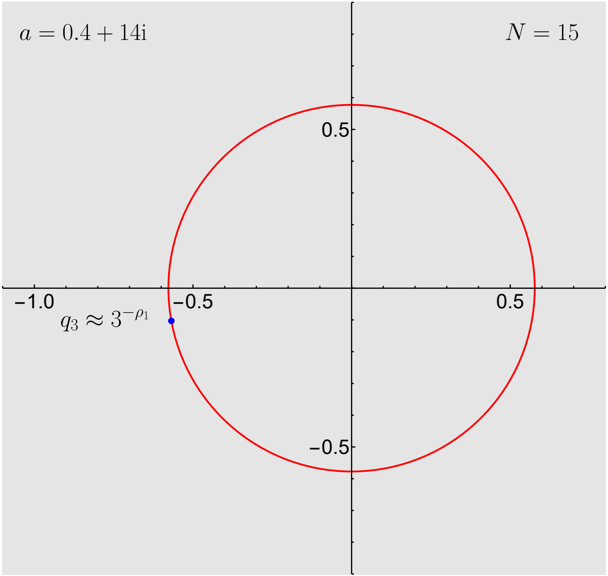
Between two zeros



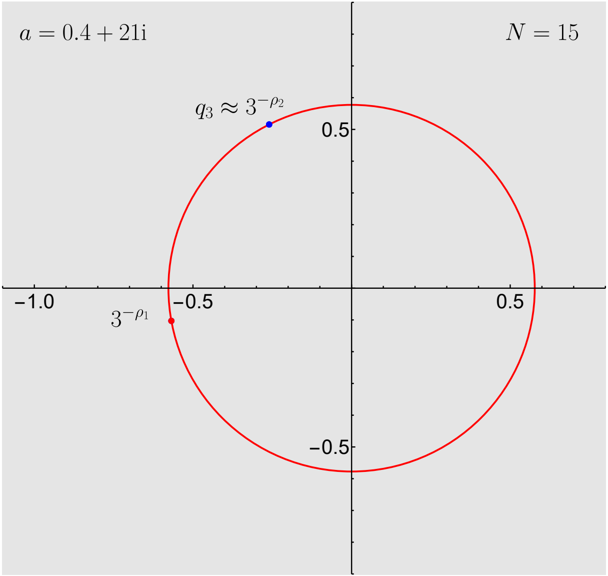
Between two zeros



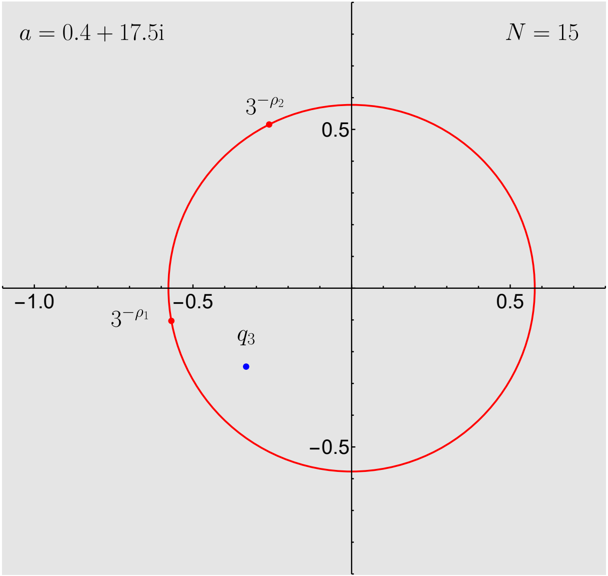
Between two zeros



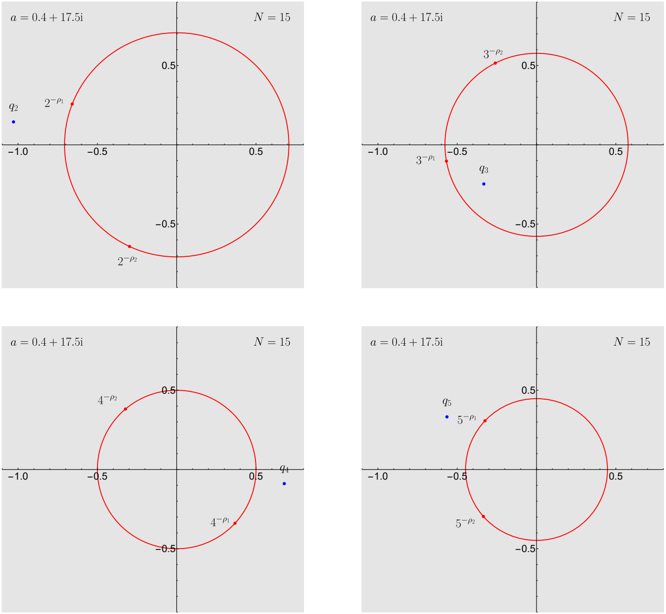
Between two zeros



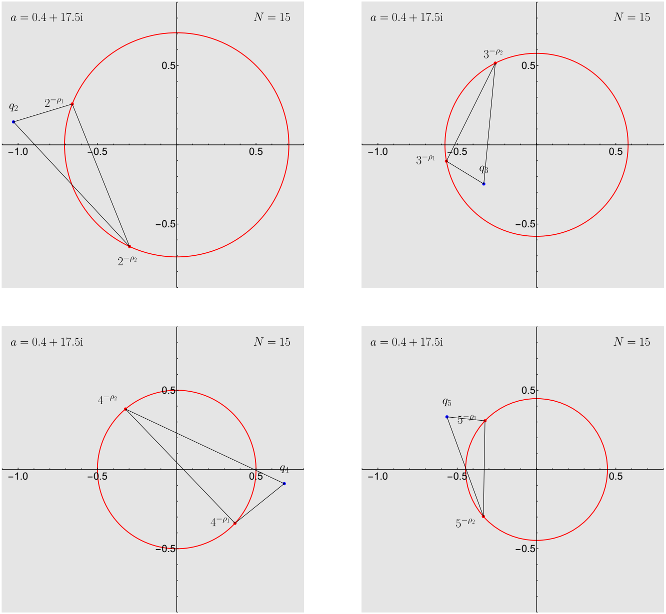
Between two zeros



Between two zeros



Between two zeros



Method 5. The similarity of the triangles

$$\frac{q_{n_2} - {n_2}^{-\rho_k}}{q_{n_1} - {n_1}^{-\rho_k}} \approx \frac{{n_2}^{-\rho_k} - {n_2}^{-\rho_{k+1}}}{{n_1}^{-\rho_k} - {n_1}^{-\rho_{k+1}}}$$

$$\left(n^2\right)^{-\rho} = \left(n^{-\rho}\right)^2 \qquad \left(n^3\right)^{-\rho} = \left(n^{-\rho}\right)^3$$

$$n^{-\rho_k} + n^{-\rho_{k+1}} \approx B/A \qquad n^{-\rho_k} n^{-\rho_{k+1}} \approx C/A$$

$$A = q_n^2 - q_{n^2} \qquad B = q_n q_{n^2} - q_{n^3} \qquad C = q_{n^2}^2 - q_n q_{n^3}$$

$n^{-\rho_k}$ and $n^{-\rho_{k+1}}$ are close to the two solutions of the equation

$$Ax^2 - Bx + C = 0$$

Numerical example

$$N = 50 \qquad a = 0.4 + 17.5 \qquad n = 2 \qquad k = 1$$

$q_2 = -0.719...+0.224...\mathrm{i}$ $\quad q_4 = 0.413...-0.283...\mathrm{i}$ $\quad q_8 = -0.166..+0.280...\mathrm{i}$

$A = 0.054...-0.039...\mathrm{i}$ $\qquad B = -0.067...+0.016...\mathrm{i}$ $\quad C = 0.033...+0.004...\mathrm{i}$

The solutions of $Ax^2 - Bx + C = 0$ are

$$x_1 = -0.65857071153... + 0.25745799250...\mathrm{i}$$

$$x_2 = \; 0.29746941585... - 0.64149196926...\mathrm{i}$$

$$\frac{x_1}{2^{-\rho_1}} = 1 + (5.39119624... - 5.16170561...\mathrm{i}) \times 10^{-18}$$

$$\frac{x_2}{2^{-\rho_2}} = 1 + (0.449985441... - 6.150248171...\mathrm{i}) \times 10^{-17}$$