Computation of solutions to certain nonlinear systems of fuzzy relation inequations

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Linear systems

Linear systems of **fuzzy relation inequalities** (**FRIs**) and **fuzzy relation equations** (**FREs**) are of the form:

$$\begin{aligned} \rho\varphi \le \psi, &\quad \varphi\rho \le \psi \\ \psi \le \rho\varphi, &\quad \psi \le \varphi\rho \\ \rho\varphi = \psi, &\quad \varphi\rho = \psi. \end{aligned}$$

where ρ is a fuzzy relation, ψ is a fuzzy relation or a fuzzy set, and φ an *unknown* fuzzy relation or a fuzzy set.

Applications

- **Addition-min** and **max-min** FRIs: BitTorrent-like Peer-to-Peer file sharing networks;
- **Max-product** FRIs: Wireless communication station networks;
- **Min-product** FRIs: Supply chain networks (eg. foodstuff supply).

The solvability of linear systems

- Linear systems may not be always **solvable**;
- Even when they are, it is not reasonable to disregard fuzzy relations which are "close enough" to exact solutions in practical situations.

Approximate solutions

- Fuzzy relations which are solutions to linear FRIs and FREs to a certain degree.
- Approximate solutions to FREs: Pedrycz [FSS, 1983], Wu [FSS, 1986], Gottwald [FSS, 1985; FSS 1994].
- More recent developments: Wu et al. [FSS, 2021], Xiao et al. [IEEE Access, 2021], Yang [Complexity, 2019].

Weakly linear systems

Weakly linear systems (**WLS**) of **FRIs** and **FREs** are of the form:

$$\varphi \circ \varrho_i \leq \varrho_i \circ \varphi, \quad \varrho_i \circ \varphi \leq \varphi \circ \varrho_i$$
$$\varrho_i \circ \varphi = \varphi \circ \varrho_i, \quad i \in I.$$

where φ is a unknown fuzzy relation and $\{\varrho_i\}_{i\in I}$- a family of fuzzy relations on A.

Applications

- **Fuzzy automata**: State reduction (right and left invariant fuzzy quasi-orders and fuzzy equivalences);
- **Fuzzy automata**: Simulations and bisimulations (forward and backward (bi)simulations);
- **Fuzzy social network analysis**: Regular fuzzy equivalences.

The solvability of weakly linear systems

- Weakly linear systems are always solvable;
- Trivial solution: The identity relation I is irrelevant;
- There are many situations where *only* the trivial solution exists.

Approximate solutions

- Used only on narrower structure, *complete Heyting algebra*.
- Our goal: Systematic approach to find approximate solutions to WLS over complete residuated lattices.

Complete residuated lattices

- **Residuated lattice:** $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$; $\wedge, \vee, \otimes, \rightarrow : L \times L \rightarrow L$; $0, 1 \in L$ (support set, meet, join, multiplication, residuum, zero, one) such that:
  - $(L, \wedge, \vee, 0, 1)$ is a *bounded distributive lattice*;
  - $(L, \otimes, 1)$ is a *commutative monoid with the unit* 1;
  - *residuation property* holds: $x \otimes y \leqslant z$ iff $x \leqslant y \rightarrow z$, for every $x, y, z \in L$.
- **Complete residuated lattice**: $(L, \wedge, \vee)$ is a complete lattice.
- **BL-algebra**: residuated lattice which satisfies prelinearity and divisibility condition.
- *Biresiduum*: $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$, for every $x, y \in L$.

Prominent examples

- $L = [0, 1]$, $\wedge = \min$, $\vee = \max$ and:

Structure	$a \otimes b$	$a \rightarrow b$
Product	$a \cdot b$	$1 (a \leq b), \quad b/a (\text{otherwise})$
Gödel	$\min\{a, b\}$	$1 (a \leq b), \quad b (\text{otherwise})$
Łukasiewicz	$\max\{a + b - 1, 0\}$	$\min\{1 - a + b, 1\}$

Fuzzy sets and fuzzy relations

$\mathcal{L}$ - a complete residuated lattice; U - a nonempty set (a universe);

- Fuzzy set over U and $\mathcal{L}$ - every mapping $\alpha : U \to L$,
- Fuzzy relation over U and $\mathcal{L}$ - every mapping $\alpha : U \times U \to L$,
- Fuzzy relation is a fuzzy set over $U \times U$ and $\mathcal{L}$.
- For fuzzy relations α and β, their **composition** is defined as

$$(\alpha \circ \beta)(u_1, u_2) = \bigvee_{u_3 \in U} \alpha(u_1, u_3) \otimes \beta(u_3, u_2), \quad \text{for every } u_1, u_2 \in U.$$

Degree of subsethood and degree of equivalence

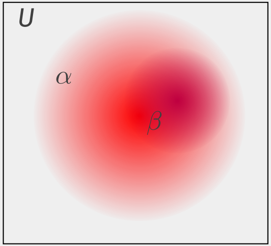
α, β - fuzzy sets (or fuzzy relations) over U and $\mathcal{L}$.

Degree of subsethood

$$S(\alpha, \beta) = \bigwedge_{u \in U} \alpha(u) \to \beta(u)$$

Degree of equivalence

$$E(\alpha, \beta) = \bigwedge_{u \in U} \alpha(u) \leftrightarrow \beta(u)$$



Instead of classical WLS- obeserving WLS similar to the given degree

$\mathcal{L}$ - a complete residuated lattice; U - a nonempty set (a universe);
$\{\varrho_i\}_{i\in I}$- a family of fuzzy relations on A; φ - an *unknown* fuzzy relation over U and $\mathcal{L}$;
λ- a scalar from $\mathcal{L}$, which presents the degree of similarity

$$
\begin{array}{llcll}
(1) & \varphi \circ \varrho_i \leq \varrho_i \circ \varphi, \quad i \in I & \rightarrow & S(\varphi \circ \varrho_i, \varrho_i \circ \varphi) \geq \lambda, \quad i \in I & (1^*) \\
(2) & \varrho_i \circ \varphi \leq \varphi \circ \varrho_i, \quad i \in I & \rightarrow & S(\varrho_i \circ \varphi, \varphi \circ \varrho_i) \geq \lambda, \quad i \in I & (2^*) \\
(3) & \varrho_i \circ \varphi = \varphi \circ \varrho_i, \quad i \in I & \rightarrow & E(\varrho_i \circ \varphi, \varphi \circ \varrho_i) \geq \lambda, \quad i \in I & (3^*)
\end{array}
$$

In the case $\lambda = 1$, $(1^*) = (1)$, $(2^*) = (2)$ and $(3^*) = (3)$.

Advantage of this approach

For most WLS, solutions that exist are not applicable for factorization and reduction

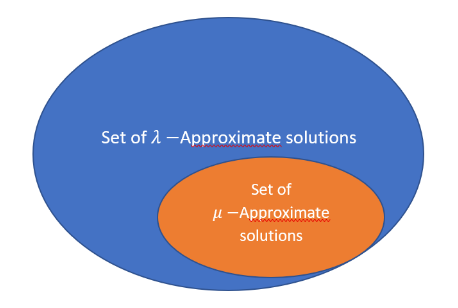
Approximation solutions form much wider class, appropriate for further use

Theorem

For every $\lambda \in \mathcal{L}$, the family of all solutions to systems (1), (2*) and (3*) form the complete lattice.*

There exists the greatest λ-approximate solution to systems (1), (2) and (3).

Class of λ and μ- approximate solutions (where $\lambda \leq \mu$)



The greatest λ-approximate solution is greater than the greatest μ-approximate solution

How to compute the greatest approximate solution?

Residuation property for fuzzy relations

- (1*) : $S(\varphi \circ \varrho, \varrho \circ \varphi) \geq \lambda$ iff $\varphi \leq (\lambda \to \varrho \circ \varphi)/\varrho$.;

  $$(\lambda \to \varrho \circ \varphi)/\varrho(u_1, u_2) = \bigwedge_{u' \in U} \varrho(u', u_1) \to (\lambda \to \varrho \circ \varphi)(u', u_2)$$

- (2*) : $S(\varrho_i \circ \varphi, \varphi \circ \varrho_i) \geq \lambda$ iff $\varphi \leq \varrho \backslash (\lambda \to \varphi \circ \varrho)$;

  $$(\varphi \circ \varrho) \backslash (\lambda \to \varphi \circ \varrho)(u_1, u_2) = \bigwedge_{u' \in U} \varphi \circ \varrho(u_2, u') \to (\lambda \to \varphi \circ \varrho)(u_1, u').$$

The procedure

$$\varphi_0 - \text{ the starting fuzzy relation,}$$

$$\varphi_{n+1} = \varphi_n \wedge \bigwedge_{i \in I} (\lambda \to \varrho_i \circ \varphi_n)/\varrho_i \wedge \bigwedge_{i \in I} \varrho_i \backslash (\lambda \to \varphi_n \circ \varrho_i), \quad n \in \mathbb{N}_0.$$

- φ_k is the greatest approximate solution to (3) **if and only** if $\varphi_k = \varphi_{k+1}$;
- If $\mathcal{L}(\{\varrho\}_{i \in I}, \lambda)$ is a finite subalgebra of $\mathcal{L}$, procedure terminates in the finite number of steps.

Theorem

φ_0 - *a fuzzy relation on A,* $\lambda \in L$, $\mathcal{L}$ - *a BL-algebra over the interval* $[0,1]$.

The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ *of fuzzy relations on A, is convergent.*

$\widetilde{\varphi} = \lim_{n\to\infty} \varphi_n$ - *is the greatest solution to (3*) contained in* φ_0.

An example

Let $\mathcal{L}$ be a product structure, the fuzzy relations $\{\varrho\}_{i\in I} = \varrho$ where:

$$\varrho = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.8 & 0 & 0.3 & 0 & 0.2 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0.4 \\ 0 & 0 & 0.8 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & 1 & 0.2 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0.1 \end{bmatrix}.$$

In the case $\lambda = 1$, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is infinite.

Example cont.

However, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is convergent:

$$\varphi = \lim_{n\to\infty} \varphi_n = \begin{bmatrix} 1 & 50/81 & 400/729 & 50/81 & 5/9 & 1600/6561 \\ 0 & 1 & 64/81 & 32/81 & 8/81 & 32/81 \\ 0 & 8/9 & 1 & 2/5 & 1/10 & 2/5 \\ 0 & 8/9 & 4/5 & 1 & 1/5 & 32/81 \\ 0 & 1 & 80/81 & 1 & 1 & 320/729 \\ 0 & 1 & 9/10 & 9/20 & 1/5 & 1 \end{bmatrix}.$$

In the case $\lambda = 0.7$

$$\varphi = \begin{bmatrix} 1 & 1 & 1 & 1 & 50/63 & 40/63 \\ 1 & 1 & 1 & 1 & 5/7 & 4/7 \\ 1 & 1 & 1 & 1 & 50/63 & 40/63 \\ 1 & 1 & 1 & 1 & 5/7 & 4/7 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Crisp part of fuzzy relation

For a fuzzy relation $\varphi \in L^{A\times A}$, the crisp part of φ is the crisp set $\varphi^c \in 2^{A\times A}$ defined by:

$$\varphi^c(a,b) = \begin{cases} 1, & \text{if } \varphi(a,b) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

for every $a, b \in A$.

Computing the greatest crisp solution

$$\varphi_0 - \text{ the starting cisp relation,}$$

$$\varphi_{n+1} = \varphi_n \wedge \bigwedge_{i\in I}((\lambda \to \varrho_i \circ \varphi_n)/\varrho_i)^c \wedge \bigwedge_{i\in I}(\varrho_i\backslash(\lambda \to \varphi_n \circ \varrho_i))^c, \quad n \in \mathbb{N}_0.$$

- φ_k is the greatest crisp solution to (3*) **if and only** if $\varphi_k = \varphi_{k+1}$;
- procedure always terminates in the finite number of steps.

An example

Let $\mathcal{L}$ be a product structure, $\{\varrho_i\}_{i\in I} = \{\varrho_1, \varrho_2\}$ are given by:

$$\varrho_1 = \begin{bmatrix} 1 & 0.95 & 0.9 & 1 & 0.9 & 0.9 & 0.8 & 0.8 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 1 & 0.4 & 0 & 0 & 0.95 & 0.4 \\ 0 & 0.95 & 0 & 0 & 0.4 & 0 & 0.9 & 0.95 \\ 0.8 & 0.95 & 0.9 & 0.95 & 0.95 & 0.8 & 1 & 0.4 \\ 0.4 & 1 & 0.6 & 0 & 0.6 & 0.6 & 0.9 & 0.9 \\ 0 & 0.95 & 0.4 & 0.9 & 1 & 0.9 & 0.8 & 0.95 \\ 0 & 0.4 & 1 & 0.8 & 0.9 & 1 & 0.95 & 0 \end{bmatrix},$$

$$\varrho_2 = \begin{bmatrix} 0.8 & 0.95 & 1 & 0.9 & 0.95 & 0.9 & 0 & 0.9 \\ 0.95 & 0.9 & 0 & 0.6 & 0 & 1 & 0.9 & 0 \\ 0.4 & 1 & 0.9 & 0.95 & 0.8 & 0 & 0.4 & 0.9 \\ 0.8 & 0.6 & 1 & 0.9 & 0.9 & 0.6 & 0.6 & 0.95 \\ 0.8 & 0.6 & 0 & 0.4 & 0.4 & 0 & 0.8 & 0.95 \\ 0 & 0.6 & 0.8 & 0.4 & 1 & 0 & 0.9 & 1 \\ 0.6 & 0 & 0.95 & 0.8 & 0.4 & 0 & 0.4 & 0.8 \\ 1 & 0 & 0.95 & 0.8 & 0.8 & 0 & 0 & 0.8 \end{bmatrix}.$$

Example cont.

Take $\lambda = 0.9$. We have infinite convergent sequence:

$$\varphi = \lim_{n\to\infty} \varphi_n = \begin{bmatrix} 1 & 1 & 0 & 40/57 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2/3 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 20/27 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Example cont.

If we employ the procedure for computing the greatest crisp solution, it ends in 3 iterations.

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Thank You!