A formal algebraic approach for the quantitative modeling of connectors in architectures

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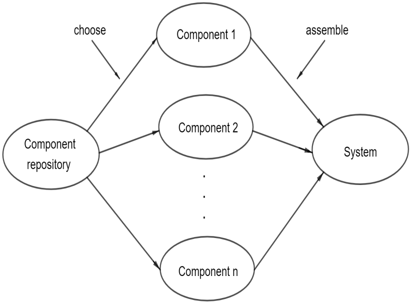
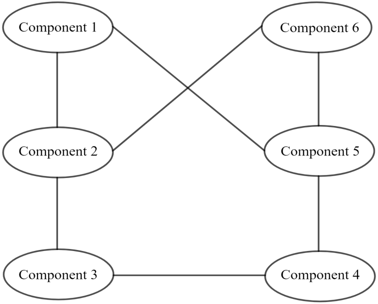
- **Component-based design** lies in the construction of **multiple components** that **interact** in order to model the global system.





- **Architectures** enforce design rules on the systems by characterizing the **topology** and the **communication patterns** of their components.

- **Coordination schemes** specify the **permissible interactions** among components.
  - Rendezvous scheme
  - Broadcast scheme

- **Connectors** regulate the **synchronization** mode among the permissible interactions.
  - All components are synchronized.
  - A component initiates the communication with the rest components.

- Connectors have been studied in several frameworks and with alternative expressiveness.

Problem to be solved

What is the problem to be solved?

↓

Model the quantitative aspects of connectors

↓

Algebraic approach in the weighted setup

- **We extend in the weighted setup the results of**:

  S. Bliudze, J. Sifakis, The algebra of connectors - Structuring interaction in BIP, (2008).

- We study an algebraic weighted framework over a **commutative** and **idempotent** semiring.

  (1) Weighted Algebra of Interactions

  (2) Weighted Algebra of Connectors

  (3) Replacing weighted connectors

  (4) Open problems

- Component-based systems consist of a finite number of components.
- We study the **coordination** aspects among the components.
- Components are connected through their **ports**.
- **Communication** among components is described by **interactions**.
- We assign to each port a **weight**.

- P: finite non-empty set of ports
- **Interaction** a: $a \in 2^P$
- **Interactions set** γ: $\gamma \in 2^{2^P}$
- Let $\Gamma(P) = 2^{2^P}$
- $(\boldsymbol{K}, +, \cdot, \hat{\mathbf{0}}, \hat{\mathbf{1}})$: commutative and idempotent semiring denoted by $\boldsymbol{K}$
- **Assign** to each port $p \in P$ a **unique weight** $k_p \in K$

Definition (Syntax)

- *P set of ports*
- *The syntax of the* **weighted Algebra of Interactions** *($wAI(P)$ for short) over P and K is given by*

$$z ::= 0 \mid 1 \mid p \mid z \oplus z \mid z \otimes z \mid (z)$$

where

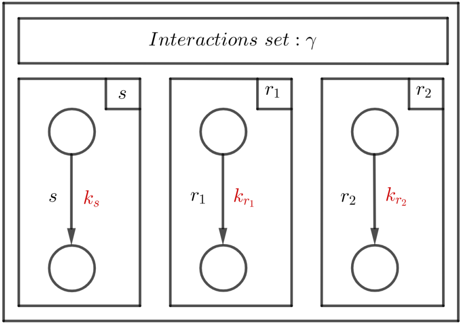
  - $0, 1$ *new symbols,*
  - $p \in P$,
  - *"$\oplus$" is the* **weighted union operator**, *and*
  - *"$\otimes$" the* **weighted synchronization operator**.

- We call z a $wAI(P)$ **element**.

Definition (Semantics)

Let z be a $wAI(P)$ element over P and K. The semantics of z is a polynomial $\|z\| \in K\langle\Gamma(P)\rangle$. For every interactions set $\gamma \in \Gamma(P)$, the value $\|z\|(\gamma)$ is defined inductively on z as follows:

- $\|0\|(\gamma) = \hat{0}$,
- $\|1\|(\gamma) = \begin{cases} \hat{1} & \text{if } \emptyset \in \gamma \\ \hat{0} & \text{otherwise} \end{cases}$,
- $\|p\|(\gamma) = \begin{cases} k_p & \text{if } \exists\, a \in \gamma \text{ such that } p \in a \\ \hat{0} & \text{otherwise} \end{cases}$,
- $\|z_1 \oplus z_2\|(\gamma) = \sum_{a\in\gamma} \left(\|z_1\|(\{a\}) + \|z_2\|(\{a\})\right)$,
- $\|z_1 \otimes z_2\|(\gamma) = \sum_{a\in\gamma} \left(\sum_{a=a_1\cup a_2} \left(\|z_1\|(\{a_1\}) \cdot \|z_2\|(\{a_2\})\right)\right)$,
- $\|(z)\|(\gamma) = \|z\|(\gamma)$.



- Apply the **Rendezvous scheme**.
- It requires the simultaneous interaction of all the components.
- $\gamma = \{\{s, r_1, r_2\}\}$

- Apply **Rendezvous** scheme.
- There is a single interaction: $\{s, r_1, r_2\}$.
- The $\boldsymbol{wAI(P)}$ **element** is:

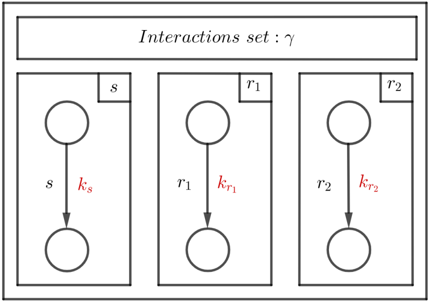
$$\boldsymbol{s \otimes r_1 \otimes r_2}$$

- For $\gamma = \{\{\boldsymbol{s}, \boldsymbol{r_2}\}\}$ we get $\| s \otimes r_1 \otimes r_2 \| (\gamma) = \hat{\mathbf{0}}$
- For $\gamma = \{\{\boldsymbol{s}, \boldsymbol{r_1}, \boldsymbol{r_2}\}\}$ we get $\| s \otimes r_1 \otimes r_2 \| (\gamma) = \boldsymbol{k_s \cdot k_{r_1} \cdot k_{r_2}}$
- For $\gamma = \{\{\boldsymbol{s}, \boldsymbol{r_1}, \boldsymbol{r_2}\}, \{\boldsymbol{s}, \boldsymbol{r_2}\}\}$ we get $\| s \otimes r_1 \otimes r_2 \| (\gamma) = \boldsymbol{k_s \cdot k_{r_1} \cdot k_{r_2}}$

- Apply the **Broadcast scheme**.

- It allows all interactions involving the sender and any subset of receivers.

- There are four interactions: $\{s\}$, $\{s, r_1\}$, $\{s, r_2\}$ and $\{s, r_1, r_2\}$.



- The ***wAI***$(\boldsymbol{P})$ **element** is:

$$\boldsymbol{s \otimes (1 \oplus r_1) \otimes (1 \oplus r_2)}$$

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$$\|s \otimes (1 \oplus r_1) \otimes (1 \oplus r_2)\| (\gamma) = \boldsymbol{k_s} + (\boldsymbol{k_s} \cdot \boldsymbol{k_{r_1}}) + (\boldsymbol{k_s} \cdot \boldsymbol{k_{r_2}}) + (\boldsymbol{k_s} \cdot \boldsymbol{k_{r_1}} \cdot \boldsymbol{k_{r_2}})$$

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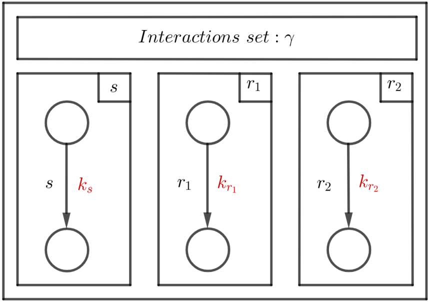

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- Components communicate through their **ports**.
- The allowed interactions are defined by the **coordination scheme**.
- **Connectors** specify the **synchronization** constraints among these interactions by relating a set of **typed ports**.
- **Types** extend ports with **synchronization modes**, specifically, with **Rendezvous** (synchron type) and **Broadcast mode** (trigger type).

Definition (Syntax)

- *P set of ports*
- *The syntax of the weighted Algebra of Connectors ($wAC(P)$ for short) over P and K is given by*

$$\sigma ::= [0] \mid [1] \mid [p] \mid [\zeta] \quad (synchron)$$

$$\tau ::= [0]' \mid [1]' \mid [p]' \mid [\zeta]' \quad (trigger)$$

$$\zeta ::= \sigma \mid \tau \mid \zeta \oplus \zeta \mid \zeta \otimes \zeta$$

where

- $0, 1$ *new symbols and* $p \in P$,
- "$\oplus$" *denotes the* **weighted union operator**,
- "$\otimes$" *denotes the* **weighted fusion operator**,
- "$[\cdot]$", "$[\cdot]'$" *are the synchron and trigger* **typing operators**.

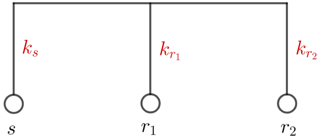
- We write $[\zeta]^{\alpha}$ for $\alpha \in \{0, 1\}$ to denote a **typed** $wAC(P)$ connector:
  - when $\alpha = 0$ it is a synchron
  - when $\alpha = 1$ it is a trigger

Definition (Semantics)

Let ζ be a $wAC(P)$ connector over P and K. The semantics of ζ is a $wAI(P)$ element defined by the function $|\cdot| : wAC(P) \to wAI(P)$ as follows:

- $\big|\, [p] \,\big| = p$, for $p \in P \cup \{0,1\}$,
- $\big|\, [p]' \,\big| = p$, for $p \in P \cup \{0,1\}$,
- $|[\zeta]| = |\zeta|$,
- $\big|\, [\zeta]' \,\big| = |\zeta|$,
- $|\zeta_1 \oplus \zeta_2| = |\zeta_1| \oplus |\zeta_2|$,
- $\big|\, [\zeta_1] \otimes [\zeta_2] \,\big| = |\zeta_1| \otimes |\zeta_2|$,
- $\big|\, [\zeta_1]^{\alpha_1} \otimes \ldots \otimes [\zeta_n]^{\alpha_n} \,\big| = \bigoplus_{\substack{i \in \{1,\ldots,n\}, \\ \alpha_i = 1}} \Big(|\zeta_i| \otimes \bigotimes_{\substack{k \neq i, \\ \alpha_k \in \{0,1\}}} (1 \oplus |\zeta_k|) \Big)$, where at least one of $\alpha_1, \ldots, \alpha_n \in \{0,1\}$ takes the value 1.

- For the examples, we omit brackets from $0, 1$ and ports $p \in P$.



- The involved components are strongly synchronized.
- The $wAC(P)$ **connector** is:

$$[s] \otimes [r_1] \otimes [r_2]$$

- The respective $wAI(P)$ **element** is:

$$\big|[s] \otimes [r_1] \otimes [r_2]\big| = |s| \otimes |r_1| \otimes |r_2| = s \otimes r_1 \otimes r_2$$

- Then the **weight** of the $wAI(P)$ element on $\gamma = \{\{s, r_1, r_2\}\}$ is

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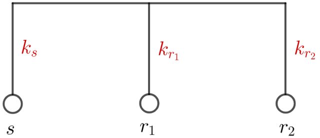

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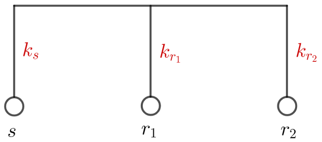

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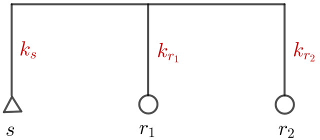
- A component triggers the interaction with some of the rest components.



- The ***wAC(P)* connector** is:

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- The respective ***wAI(P)* element** is:

$$\begin{aligned} \left\| [s]' \otimes [r_1] \otimes [r_2] \right\| &= |s| \otimes (1 \oplus |r_1|) \otimes (1 \oplus |r_2|) \\ &= s \otimes (1 \oplus r_1) \otimes (1 \oplus r_2) \end{aligned}$$

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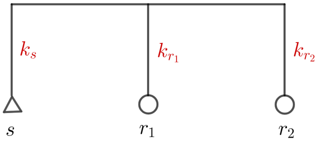

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- Then the **weight** of the $wAI(P)$ element on $\gamma = \{\{\boldsymbol{s}\}, \{\boldsymbol{s}, \boldsymbol{r_1}\}, \{\boldsymbol{s}, \boldsymbol{r_2}\}, \{\boldsymbol{s}, \boldsymbol{r_1}, \boldsymbol{r_2}\}\}$ is

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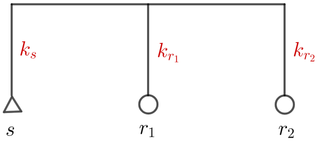

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- $z_1, z_2 \in wAI(P)$ are **equivalent** $z_1 \equiv z_2$, when

$$\|z_1\|(\gamma) = \|z_2\|(\gamma) \text{ for every } \gamma \in \Gamma(P).$$

- $\zeta_1, \zeta_2 \in wAC(P)$ are **equivalent** $\zeta_1 \equiv \zeta_2$, when

$$|\zeta_1| = |\zeta_2|.$$

- Congruence relation is important because it allows to replace a $wAC(P)$ connector with another one.

- **Congruence relation:**
    - Given two equivalent elements, if we apply any operator, then we obtain again two equivalent elements.

- The weighted fusion operator requires specifying a typing.

- **Restrict on fusion-$wAC(P)$ connectors** which are typed by definition.

- $\zeta = [\zeta_1]^{\alpha_1} \otimes \ldots \otimes [\zeta_n]^{\alpha_n} \in wAC(P)$ is a **fusion-$wAC(P)$ connector.**

- Weighted fusion operator does not preserve the equivalence of fusion-$wAC(P)$ connectors.

Definition

We denote by "$\cong$" the largest "congruence relation" for fusion-$wAC(P)$ *connectors, contained in $\equiv$ of $wAC(P)$, i.e., the largest relation satisfying the following:*

For **fusion-$wAC(P)$** *connectors* ζ_1, ζ_2 and $r \notin P$,

$$\zeta_1 \cong \zeta_2 \Rightarrow \forall E \in wAC\left(P \cup \{r\}\right), E(\zeta_1/r) \equiv E(\zeta_2/r)$$

where $E(\zeta/r)$ denotes the expression obtained from E by replacing all occurrences of $wAC(P)$ connector r by ζ.

Theorem 1

Let ζ_1, ζ_2 be fusion-$wAC(P)$ *connectors. Then*

$$\zeta_1 \equiv \zeta_2 \Leftrightarrow [\zeta_1]^\alpha \cong [\zeta_2]^\alpha \text{, for any } \alpha \in \{0, 1\}.$$

- **Degree** of a fusion-$wAC(P)$ connector ζ is the number of its trigger elements denoted by $\#_T\zeta$.

Theorem 2

Let ζ_1, ζ_2 be fusion-$wAC(P)$ *connectors. Then*

$$\zeta_1 \cong \zeta_2 \Leftrightarrow \begin{cases} \zeta_1 \equiv \zeta_2 \\ \zeta_1 \otimes [1]' \equiv \zeta_2 \otimes [1]' \\ \#_T\zeta_1 > 0 \Leftrightarrow \#_T\zeta_2 > 0. \end{cases}$$

Future work directions

- Investigate a real congruence relation for any $wAC(P)$ connector.
- Study the computation and coordination of component-based systems and their connectors in our weighted setup.
- Extend the results for weighted connectors with dynamic interactions.

Thank you for your attention!